# STUDY OF SOME PROPERTIES OF KONHAUSER BIORTHOGONAL POLYNOMIALS

THESIS PRESENTED

BY

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TO FULFILL THE PARTIAL REQUIREMENT FOR THE DEGREE OF

**MASTER OF PHILOSOPHY**

OF

BUNDELKHAND UNIVERSITY
JHANSI (INDIA)

Department of
Mathematics and Statistics

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1989

CERTIFICATE

This is to certify that the work embodied in this thesis entitled " STUDY OF SOME PROPERTIES OF KONHAUSER BIORTHOGONAL POLYNOMIALS " being submitted by Rajendra Kumar Agarwal, to fulfill the partial requirement for the degree of M. Phil. of Bundelkhand University, Jhansi U.P. is upto the mark, both in academic contents and quality of presentation. I further certify that this work has been done by him under my supervision and guidance.

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# PREFACE

The present work is outcome of the studies done by me in the field of Special Functions, with special emphasis on " some properties of Konhauser Biorthogonal Polynomials " at the department of Mathematics and Statistics, Bundelkhand University, Jhansi during the course of studies for the degree of M.Phil.

The present work has been done under the able guidance of Dr. P.N. Shrivastava, Reader and Head of the Mathematics Department, Bundelkhand University, Jhansi.

I express my deepest sense of gratitute to Dr. P.N. Shrivastava for competent guidance and unbounding interest in the preparation of this thesis. I am also thankful to Dr. V.K. Sehgal and Dr. V.K. Singh of the department for their continuous encouragement. I am also indebted to my parents for their continuous interest to fulfill my goal.

This thesis consists of four Chapters each divided into several sections (Progressively 1.1, 1.2,.........). The formulae are numbered progressively within each section for instance (3.6.1) denotes the Ist formula in 6th article of 3rd Chapter. After the Chapters, there is an Appendix, which includes those theorems and formulas which have been used in the thesis. References are given at the end of the thesis in alphabatical order. In the thesis by [5] we mean the reference given at number 5 in the list of references

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## CONTENTS

# CHAPTER-I

# INTRODUCTION

## 1.1 BIORTHOGONAL SYSTEMS:

Let $\alpha(x)$ be a distribution function on the interval (finite or infinite) $[a,b]$ with infinitely many points of increase and such that

$$\int_a^b x^n \, d\alpha(x) < \infty \quad \text{for all} \quad n = 0,1,2,\ldots\ldots$$

The set of polynomials in $x\{P_n(x)\}$, and the set of polynomials $\{Q_n(x)\}$ deg. $Q_n(x) = n$, $n = 0,1,2,\ldots\ldots$ are said to be Biorthogonal with respect to $d\alpha(x)$, on $[a,b]$ if

$$(1.1.1) \qquad \int_a^b P_n(x)\, Q_m(x)\, d\alpha(x) \begin{cases} = 0, & m \neq n \\ \neq 0, & m = n \end{cases}$$

Didon [15] and Deruyts [14] considered this concept in some detail. For example for a given $\{P_n(x)\}$, the set $\{Q_n(x)\}$ is uniquely determined and conversely-Presier [24] and Konhauser [19] reconsidered this concept. It is shown that (1.1.1) is equavalent to (1.1.2) and (1.1.3)

$$(1.1.2) \qquad \int_a^b x^i\, P_n(x)\, d\alpha(x) \begin{cases} = 0, & 0 \leq i \leq n \\ \neq 0, & i = n \end{cases}$$

and

$$(1.1.3) \qquad \int_a^b x^{ik} Q_n(x)\, d\alpha(x) \quad \begin{cases} = 0 , & 0 \le i < n \\ \ne 0 , & i = n \end{cases}$$

Thus if k = 1, $\{P_n(x)\}$ and $\{Q_n(x)\}$ collapse to the set of orthogonal polynomials associated with $\alpha(x)$ on (a,b) Both Didon and Deruyts gave as an example

The case $d\alpha(x) = x^{\alpha-1}(1-x)^{\beta-1}$ on (0,1).

More recently these polynomials gained sudden popularity with the interesting work of spencer and Fano [31] Konhauser [19,20] Presier [24] , Carlitz [5] , Chai [7] etc.

In particular, the biorthogonal system associated with Laguerre distribution is known as Konhauser Biorthogonal polynomials.

1.2 <u>SPENCER AND FANO POLYNOMIALS:</u> In 1951, Spencer and Fano [31] in certain calculations involving penetration of Gamma rays through matter, encountered with a pair of polynomials $Z_n^{(\Omega)}(x)$ and $y_n^{(\Omega)}(x)$, which are solutions of the following third order differential equations respectively:

$$(1.2.1) \qquad x Z_l''' + (1+\Omega-3x) Z_l'' + 2(x-1-\Omega) Z_l' = 2 Z_l$$

and

$$(1.2.2) \qquad x y_n''' + 2(1+\Omega) y_n'' + \left[\frac{\Omega(1+\Omega)}{x} - x\right] y_n' = -2n y ,$$

where $\Omega$ is a parameter with $Re(\Omega) > -1$.

The integral representations for these polynomials are

$$(1.2.3)\quad Z_{l}^{(\alpha)}(x) = \frac{(-1)^{l+\alpha}}{\pi i}\int_{C}\frac{e^{xt}(1-t)^{\alpha+2l}}{[t(t-2)]^{l+1}}\,dt\,,$$

where C is a closed contour enclosing t = 0, but excluding t = 1, 2, as long as 1 is an integer.

$$(1.2.4)\quad y_{n}^{(\alpha)}(x) = \frac{x^{-\alpha}}{2\pi i}\int_{C'}\frac{e^{xt}\,[t(t+2)]^{n}}{(t+1)^{2n+1+\alpha}}\,dt\,,$$

where C' encircles t = $-1$, n and $\alpha$ are integrals. If we move C' such that t + 1 = s, then C" is contour encircling s = 0, and then

$$(1.2.5)\quad y_{n}^{\alpha}(x) = \frac{x^{-\alpha}}{2\pi i}\int_{C''}\frac{e^{xs}(s^{2}-1)^{n}}{s^{2n+1+\alpha}}\,ds\,.$$

From (1.2.3) and (1.2.5), we see that

$$(1.2.6)\quad Z_{l}^{\alpha}(x) = \frac{(-1)^{\alpha}}{2^{l}\,l!}\,\frac{\partial^{l}}{\partial t^{l}}\left[\frac{e^{xt}(1-t)^{\alpha+2l}}{(1-t/2)^{l+1}}\right]_{t=0}$$

and

$$(1.2.7)\quad e^{x}\,Y_{n}^{\alpha}(x) = \frac{1}{(2n+\alpha)!}\,\frac{\partial^{2n+\alpha}}{\partial s^{2n+\alpha}}\left[e^{xs}(s^{2}-1)^{n}\right]_{s=0}\,,$$

where $Y_{n}^{\alpha}(x) = x^{\alpha}e^{-x}\,y_{n}^{\alpha}(x)$.

The series expansions for $Z_n^{\Omega}$ and $Y_n^{\Omega}$ are given by

$$(1.2.8) \qquad Z_\ell^{\Omega}(x) = \sum_{j=0}^{\ell} b_{\ell j}\, x^j , \qquad \text{where}$$

$$(1.2.9) \qquad b_{\ell j} = \frac{(-1)^\ell}{2^{2\ell}} \frac{(2\ell+\Omega)!}{\ell!} \frac{2^j}{j!} \sum_{k=0}^{\ell-1} \frac{(-1)^k (2\ell-j-k)!\, 2^k}{(\Omega+2\ell-k)!\, k!\, (\ell-j-k)!} , \qquad \ell \neq j$$

and

$$b_{\ell\ell} = \frac{(-1)^\ell}{2^\ell \cdot \ell!} ;$$

$$(1.2.10) \qquad Y_n^{\Omega}(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{2j}}{(2j+\Omega)!} .$$

Interestingly, these polynomials happen to satisfy the biorthogonality relation

$$(1.2.11) \qquad \int_0^\infty x^{\Omega} e^{-x} Z_\ell^{\Omega}(x)\, Y_n^{\Omega}(x)\, dx = \delta_{\ell n} ,$$

where
$$\delta_{\ell n} = 0 , \quad \ell \neq n$$
$$= 1 , \quad \ell = n .$$

S. Prieser [24] made further investigations of the biorthogonal polynomials derivable from the ordinary differential equations of third order.

1.3 KONHAUSER POLYNOMIALS: Motivated by the works of Spencer and Fano [31] and Prieser [24], Konhauser [20] in 1967 considered a system of biorthogonal polynomials, suggested by Laguerre polynomials, Konhauser denoted these polynomials by $Z_n^c(x;k)$ and $Y_n^c(x,k)$, defined them in terms of integral representations as

$$(1.3.1) \quad Y_n^c(x,k) = \frac{k}{2\pi i} \int_C \frac{e^{-xt}(t+1)^{c+kn}}{[(t+1)^k - 1]^{n+1}} \, dt,$$

where C is a contour enclosing t = 0, but excluding t = - 1 and the roots of the equation $(t+1)^k - 1 = 0$ .

$$(1.3.2) \quad x^c Z_n^c(x;k) = \frac{1}{2\pi i} \int_{C'} \frac{e^{xs}(s^k - 1)^n}{s^{1+c+kn}} \, ds,$$

where C' is a contour enclosing s = 0.

From (1.3.1) and (1.3.2), we get $n^{th}$ differential formulas for these polynomials as

$$(1.3.3) \quad Y_n^c(x,k) = \frac{k}{n!} \frac{\partial^n}{\partial t^n} \left[ \frac{e^{-xt}(t+1)^{c+kn}}{(t^{k-1} + kt^{k-2} + \cdots\cdots + k)^{n+1}} \right]_{t=0}$$

and when C is an integer

$$(1.3.4)\quad Z_n^c(x,k) = \frac{x^{-c}}{(c+kn)!}\,\frac{\partial^{c+kn}}{\partial s^{c+kn}}\left[e^{xs}(s^k-1)^n\right]_{s=0}$$

$Z_n^c(x,k)$ and $Y_n^c(x,k)$ satisfy the following $(k+1)^{th}$ order differential equations:

$$(1.3.5)\quad D^k\left(x^{c+1} D Z_n^c(x,k)\right) = x^{c+1} D Z_n^c(x,k) - nk x^c Z_n^c(x,k)$$

and

$$(1.3.6)\quad \left\{\left[x(D-1)+c+1\right]\left[(D-1)^k-(-1)^k\right]-(-1)^k kn\right\}Y_n^c(x,k) = 0$$

The series expansions for $Z_n^c(x;k)$ and $Y_n^c(x;k)$ are given by,

$$(1.3.7)\quad Z_n^c(x;k) = \frac{\Gamma(1+c+kn)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(1+c+kj)}$$

and

$$(1.3.8)\quad Y_n^c(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{1+c+j}{k}\right)_n ,$$

where $(a)_n = a(a+1)\cdots\cdots(a+n-1),\ n \geq 1$

and $(a)_0 = 1 , \quad a \neq 0 .$

These polynomials satisfy the biorthogonality relation

$$(1.3.9) \quad \int_0^\infty x^c e^{-x} Y_i^c(x;k)\, Z_j^c(x;k)\, dx = \frac{\Gamma(1+c+kj)}{j!} \delta_{ij} ,$$

$\forall\ i, j = 0, 1, 2, \ldots\ldots$ and $\delta_{ij} = 0$ , $i \neq j$ ;
and $\delta_{ii} = 1$.

Interestingly, $Z_n^c(x,k)$ and $Y_n^c(x,k)$ reduce to the polynomials, those of Spencer and Fano [31] for k = 2 and in respective notations

$$(1.3.10) \quad Z_n^c(x,2) = y_n^c(x) \quad \text{and}$$
$$Y_n^c(x,2) = z_n^c(x) .$$

Also, for k = 1, both (1.3.7) and (1.3.8) reduce to Laguerre polynomials $L_n^\alpha(x)$ , for $\alpha = c+1$, defined by Rodrigue's formula

$$(1.3.11) \quad L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} D^n \left[ e^{-x} x^{n+\alpha} \right] .$$

The series expansion for $L_n^\alpha(x)$ is

$$(1.3.12) \quad L_n^\alpha(x) = \sum_{m=0}^{n} \frac{(-1)^m (1+\alpha)_n x^m}{m! (n-m)! (1+\alpha)_m} .$$

The series expansion (1.3.8) was given by L. Carlitz [5], using which H.M. Srivastava [33] showed that

$$(1.3.13) \quad Y_n^c(x,k) = \frac{x^{-kn-c-1}\, e^{x}}{k^n\, n!} \left(x^{k+1} D_x\right)^n \left[x^{c+1} e^{-x}\right].$$

Clearly (1.3.13) reduces to (1.3.11) for k = 1 and

However (1.3.13) has been given by Calvez et Genin [ 4 ].

## 1.4 POLYNOMIAL DEFINED BY THE GENERALISED RODRIGUES FORMULA:

As we have seen above (1.3.13), $Y_n^c(x,k)$ has been expressed by a Rodrigue's type formula. Hence it is appropriate to give a brief account of such formulas, which have been useful in subsequent chapters.

The classical orthogonal polynomials have a generalised Rodrigue's formula of the form

$$(1.4.1) \quad F_n(x) = \frac{1}{K_n\, w(x)} D^n \left[w(x) \cdot X^n\right], \quad D \equiv d/dx ,$$

where $K_n$ is a constant, X is a polynomial in x, whose coefficients are independent of n, and w ( x ) is a weight function, where as $F_n$ (x) is a polynomial of degree n is x. The Rodrigue's formulae for Laguerre and Hermite polynomials which are the particular cases of the above formula are as follows:

$$(1.4.2) \quad L_n^{\alpha}(x) = \frac{1}{n!} x^{-\alpha} e^{x} \cdot D^n \left(x^{\alpha+n} e^{-x}\right) ,$$

$$(1.4.3) \quad H_n(x) = (-1)^n e^{x^2} D^n \left(e^{-x^2}\right) .$$

In 1938 Angelescu [1] considered the polynomials $\Pi_n(x)$ connected with Appell and defined as

(1.4.4) $$\Pi_n(x) = e^{x} D^n \{ e^{-x} A_n(x) \} ,$$

where the set of polynomials $A_n(x)$ forms an Appell set.

Krall-Frink [21] in 1949 obtained a class of polynomials which they called ' Bessel polynomials '. These arise as the solution of the classical wave equation in spherical coordinates. They define them as

(1.4.5) $$y_n(x;a,b) = b^{-n} x^{2-a} e^{b/x} D^n [x^{2n+a-2} e^{-b/x}] .$$

Another interesting study, starting with Rodrigue's formula is due to E.T. Bell (1934) [3]. He considers the polynomials $\xi_n(x,t;r)$ given by

(1.4.6) $$\xi_n(x,t;r) = \exp(-xt^r) D^n ( e^{xt^r}) .$$

Maurice de Duffahel [16] in 1936 has defined and studied, is not so well known paper, polynomials $p_n(x)$ where

(1.4.7) $$p_n(x) = \frac{e^{x^2}}{n!} D^n ( x^n e^{-x^2}) .$$

Following E.T. Bell, A.M. Chak [8] in 1956, considered the polynomials $P_{n,r}^{(\alpha)}(x)$ in $x^r$, defined by the Rodrigue's formula.

(1.4.8) $$P_{n,r}^{(\alpha)}(x) = \frac{e^{x^r} x^{-\alpha}}{n!} \frac{d^n}{dx^n} [x^{n+\alpha} e^{-x^r}]$$

In 1959 F.J. Palas [23] started with the generating function

$$(1.4.9)\quad (1-t)^{-1}\exp[x^k U(t)] = \sum_{n=0}^{\infty} T_{kn}(x)\, t^n ,$$

where $U(t) = 1-(1-t)^{-k}$, and showed that the polynomials $T_{kn}$ satisfy the Rodrigue's formula:

$$(1.4.10)\quad T_{kn}(x) = \frac{e^{x^k}}{n!}\left(\frac{d}{dx}\right)^n \left(x^n e^{-x^k}\right) .$$

In 1962 Gould-Hopper [18] studied two generalisations of Hermite polynomials, viz.

$$(1.4.11)\quad H_n^{r}(x,a,p) = (-1)^n x^{-a} e^{px^r} D^n \left[x^a e^{-px^r}\right]$$

and

$$(1.4.12)\quad g_n^{r}(x,h) = e^{-hD^r} x^n .$$

During 1963 and 1964, Chatterjee [10] and Singh-Srivastava [30] proceeded simultenously to define generalisation of Laguerre polynomials. Singh-Srivastava (1963) gave the generalisation following Gould-Hopper, as

$$(1.4.13)\quad L_n^{(\alpha)}(x,r,p) = \frac{1}{n!} x^{-\alpha} e^{px^r} D^n \left[x^{\alpha+n} e^{-px^r}\right] ,$$

following the extension of Bessel polynomials given by N. Obreskov [22] (1956)

(1.4.14) $P_n^{(m)}(x) = x^{-m} e^{-1/x} D^n [x^{2n+m} e^{1/x}]$,

In 1964, Chatterjee [11, 12] gave generalised Bessel polynomials as

(1.4.15) $M_n^{(k)}(x,a,b) = b^{-n} x^{k-a-(k-2)n} e^{b/x} \cdot$
$\cdot D^n [x^{kn+a-k} e^{-b/x}]$.

Chatterjee [13] (1966) defined a function $F_n^{(r)}(x;a,k,p)$ by a generalised Rodrigue's formula as

(1.4.16) $F_n^{(r)}(x;a,k,p) = x^{-a} e^{px^r} D^n [x^{kn+a} e^{-px^r}]$.

Motivated by operators used by Carlitz, in 1956 A.M. Chak [8] defined a function $G_{n,k}^{(\alpha)}(x)$ as

(1.4.17) $G_{n,k}^{(\alpha)}(x) = x^{-\alpha-kn+n} e^{x} (x^k D)^n e^{-x} x^{\alpha}$.

Recently R.P.Singh [29] following Gould-Hopper [18] has given a generalisation to Tascano's polynomials by the relation

(1.4.18) $T_n^{(\alpha)}(x,r,p) = x^{-\alpha} e^{px^r} (xD)^n (x^{\alpha} e^{-px^r})$.

Following Singh [29] and Chak [8], R.C. Chandel [9] defined another generalised Truesdell polynomials $T_n^{(\alpha,k)}(x,r,p)$ as

(1.4.19) $T_n^{(\alpha,k)}(x,r,p) = x^{-\alpha} e^{px^r} (x^k D)^n [x^{\alpha} e^{-px^r}]$.

Simulteneously P.N. Shrivastava[26,27,28], Srivastava and Singhal[36] considered a class of polynomials defined respectively by relations:

$$(1.4.20)\quad F_n^{(r,m)}(x;a,k,p) = x^{-a} e^{px^r} (x^k D)^n \left[ x^{a+mn} e^{-px^r} \right],$$

$$(1.4.21)\quad G_n^{(\alpha)}(x,r,p,k) = \frac{x^{-kn-\alpha} \exp(px^r)}{n!} (x^{k+1} D)^n \left[ x^\alpha e^{-px^r} \right].$$

Both (1.4.20) and (1.4.21) happens to include Laguerre, Hermite and Bessel polynomials as their particular cases. The interesting point here is (1.3.13) is also a particular case of (1.4.21), related as

$$(1.4.22)\quad Y_n^\alpha(x;k) = k^{-n} G_n^{(\alpha+1)}(x,1,1,k).$$

As such many properties of $Y_n^{(\alpha)}(x,k)$ follows from those of $G_n^\alpha(x,r,p,k)$, as particular cases for r = p = 1.

## 1.5 OPERATIONAL PROPERTIES OF OPERATOR $\Theta = x^{k+1} D$ :

We mention below some well known properties of $\Theta = x^{k+1} D$:

$$(1.5.1)\quad \Theta^n x^\alpha = (\alpha)^{(k,n)} x^{\alpha+kn},$$

where $(\alpha)^{(k,n)} = \alpha(\alpha+k) \cdots\cdots (\alpha+(n-1)k)$,

$= k^n (\alpha/k)_n .$

(1.5.2) $$e^{t\theta} f(x) = f\left(\frac{x}{(1-ktx^k)^{1/k}}\right),$$

(1.5.3) $$\theta^n (U\cdot V) = \sum_{i=0}^{n} \binom{n}{i} (\theta^{n-i} U)\cdot(\theta^i V),$$

(1.5.4) $$e^{t\theta}(U\cdot V) = e^{t\theta} U \cdot e^{t\theta} V,$$

(1.5.5) $$F(\theta)(x^\alpha g(x)) = x^\alpha F(\alpha x^k + \theta) g(x),$$

(1.5.6) $$F(\theta)(e^{g(x)} \cdot f(x)) = e^{g(x)} F(x^{k+1} g'(x) + \theta) f(x),$$

and the generalised rule of differentiation as

(1.5.7) $$\theta_x^n f(z(x)) = \sum_{p=0}^{n} \frac{(-1)^p}{p!} \left(\frac{d}{dz}\right)^p f(z) \sum_{r=0}^{p} (-1)^r \binom{p}{r} z^{p-r} \theta^n z^r.$$

From (1.5.1) and (1.5.3), we shall have

(1.5.8) $$(\alpha+\beta)^{(k,n)} = \sum_{p=0}^{n} \binom{n}{p} (\alpha)^{(k,n-p)} (\beta)^{(k,p)},$$

which reduces to Binomial theorem for k = 0 .

Also we have

(1.5.9) $$(x^2 D)^n \{g(x)\} = x^{n+1} D^n \{x^{n-1} g(x)\},$$

for every non-negative integer n.

The purpose of the present thesis is to discuss in detail the following two papers of H.M. Srivastava related to Konhauser biorthogonal polynomials and their multilinear generating relations.

(A) Some Biorthogonal polynomials suggested by the Laguerre polynomials; Pac. J. Math, Vol.98(1), 1982, pp.235-250.

(B) A multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials; Pac. J. Math, Vol.117 (1),1985 pp. 183-191.

The Chapter II and III deals with the paper (A) and Chapter IV deals with the paper (B) above.

# CHAPTER-II

## PROPERTIES OF KONHAUSER POLYNOMIALS $Y_n^{\alpha}(x;k)$

### 2.1 INTRODUCTION:

We begin with recalling the polynomials $G_n^{(\alpha)}(x,r,p,k)$ which were introduced Srivastava and Singhal [36] to provide an elegant form of the various known generalisation of the classical Hermite and Laguerre polynomials. These polynomials are defined by the generalised Rodrigue's formula (1.4.21)

$$G_n^{(\alpha)}(x,r,p,k) = \frac{x^{-kn-\alpha}\exp(px^r)}{n!}\left(x^{k+1}D_x\right)^n\left\{x^{\alpha}\exp(-px^r)\right\}$$

where $D_x = d/dx$, and the parameters $\alpha$, k, p and r are unrestracted in general. The explicit expression is given by [36]

$$(2.1.1)\quad G_n^{(\alpha)}(x,r,p,k) = \frac{k^n}{n!}\sum_{i=0}^{n}\frac{(px^r)^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{rj+\alpha}{k}\right)_n .$$

On comparing (2.1.1) with Carlitz's result (1.3.8) we get known relationship (1.4.22)

$$Y_n^{\alpha}(x;k) = k^{-n}\,G_n^{(\alpha+1)}(x,1,1,k),\quad \alpha > -1,\ k = 1,2,3,\cdots$$

Hence from this relation we get Rodrigue's formula for $Y_n^{\alpha}(x;k)$ as

$$(2.1.2)\quad Y_n^{\alpha}(x;k) = \frac{x^{-kn-\alpha-1}}{k^n\, n!}\, e^{x}\left(x^{k+1}D_x\right)^n\left\{x^{\alpha+1}e^{-x}\right\}.$$

### 2.2 GENERATING RELATIONS I:

Following are the generating relations for $Y_n^{\alpha}(x;k)$:

$$(2.2.1)\quad \sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\,t^n = (1-t)^{-(\alpha+1)/k} \exp\left(x\left[1-(1-t)^{-\frac{1}{k}}\right]\right),$$

$$(2.2.2)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha}(x;k)\,t^n = (1-t)^{-(\alpha+1)/k} \exp\left(x\left[1-(1-t)^{-\frac{1}{k}}\right]\right) \cdot Y_m^{\alpha}\left(x(1-t)^{-\frac{1}{k}};k\right),$$

where m is a non-negative integer.

To prove (2.2.1) and (2.2.2) we first prove the following generating relations for $G_n^{(\alpha)}(x,r,p,k)$:

$$(2.2.3)\quad \sum_{n=0}^{\infty} t^n G_n^{(\alpha)}(x,r,p,k) = (1-kt)^{-\alpha/k} \exp\left[px^r\left(1-(1-kt)^{-\frac{r}{k}}\right)\right].$$

$$(2.2.4)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} t^n G_{m+n}^{(\alpha)}(x,r,p,k)$$
$$= (1-kt)^{-\alpha/k} \exp\left[px^r\left(1-(1-kt)^{-\frac{r}{k}}\right)\right] \cdot G_m^{(\alpha)}\left(x(1-kt)^{-\frac{1}{k}},r,p,k\right).$$

Now from equation (1.4.21), and letting $u = x^{-k}$,

$$\sum_{n=0}^{\infty} t^n G_n^{(\alpha)}(x,r,p,k) = x^{-\alpha}\exp(px^r) \sum_{n=0}^{\infty} \frac{(t\,x^{-k})^n}{n!} \left(x^{k+1}D_x\right)^n \left[x^{\alpha}\exp(-px^r)\right]$$

$$= x^{-\alpha}\exp(px^r)\left[x^{\alpha}(1-kt)^{-\alpha/k}\exp\left\{-px^r(1-kt)^{-r/k}\right\}\right]$$

[by use of (1.5.2)]

$$= (1-kt)^{-\alpha/k}\exp\left[px^r\left(1-(1-kt)^{-\frac{r}{k}}\right)\right].$$

This proves (2.2.3).

From (2.2.3)

$$\sum_{n=0}^{\infty} \left(\frac{t}{k}\right)^n G_n^{(\alpha)}(x,r,p,k) = (1-t)^{-\frac{\alpha}{k}} \exp\left[px^r\left(1-(1-t)^{-\frac{r}{k}}\right)\right].$$

Hence for r = p = 1 and $\alpha$ is replaced by $\alpha+1$ , and using (1.4.22), we get

$$\sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\, t^n = (1-t)^{-(\alpha+1)/k} \exp\left(x\left[1-(1-t)^{-\frac{1}{k}}\right]\right),$$

which proves (2.2.1) .

Now from relation (1.4.21)

$$\sum_{n=0}^{\infty} \binom{m+n}{n} t^n G_{m+n}^{(\alpha)}(x,r,p,k) = x^{-km-\alpha} \exp(px^r)(x^{k+1}D_x)^m \cdot$$
$$\cdot \sum_{n=0}^{\infty} \frac{(t x^{-k})^n}{n!} (x^{k+1}D_x)^n \left\{x^{\alpha}\exp(-px^r)\right\}$$

$$= \frac{x^{-km-\alpha} \cdot \exp(px^r)(x^{k+1}D_x)^m}{m!} \cdot \left[\left\{x(1-kt)^{-\frac{1}{k}}\right\}^{\alpha} \exp\left(-p\left(x(1-kt)\right.\right.\right.$$

[by use of (1.5.2)]

$$= \frac{x^{-km-\alpha}\exp(px^r)}{m!} \cdot m!\, x^{km+\alpha}(1-kt)^m (1-kt)^{-(km+\alpha)/k} \cdot$$
$$\cdot \exp\left[-p\left(x(1-kt)^{-\frac{1}{k}}\right)^r\right] G_m^{(\alpha)}\left(x(1-kt)^{-\frac{1}{k}}, r, p,\right.$$

$$= (1-kt)^{-\frac{\alpha}{k}} \exp\left[px^r\left(1-(1-kt)^{-\frac{r}{k}}\right)\right] \cdot G_m^{(\alpha)}\left(x(1-kt)^{-\frac{1}{k}}, r, p, k\right).$$

This proves (2.2.4).

From (2.2.4), we have

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \left(\frac{t}{k}\right)^n G_{m+n}^{(\alpha)}(x,r,p,k) = (1-t)^{-\frac{\alpha}{k}} \exp\left[px^r\left(1-(1-t)^{-\frac{r}{k}}\right)\right] \cdot$$
$$\cdot G_m^{(\alpha)}\left(x(1-t)^{-\frac{1}{k}}, r, p, k\right) .$$

Hence for r = p = 1 and $\alpha$ is replaced by $\alpha+1$ , and using (1.4.22), we get

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha}(x;k)\, t^n = (1-t)^{-(\alpha+1)/k} \exp\left(x\left[1-(1-t)^{-\frac{1}{k}}\right]\right)$$
$$\cdot Y_m^{\alpha}\left(x(1-t)^{-\frac{1}{k}}; k\right),$$

which proves (2.2.2).

2.3 GENERATING RELATIONS II :

Following are the generating relations for $Y_n^{\alpha}(x;k)$ which shall be obtained by use of Lagrange's expansion (See Appendix):

(2.3.1) $$\sum_{n=0}^{\infty} Y_n^{\alpha-kn}(x;k)\, t^n = (1+t)^{(\alpha-k+1)/k} \exp\left(x\left[1-(1+t)^{\frac{1}{k}}\right]\right),$$

(2.3.2) $$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha-kn}(x;k)\, t^n = (1+t)^{(\alpha-k+1)/k} \cdot \exp\left(x\left[1-(1+t)^{\frac{1}{k}}\right]\right) \times Y_m^{\alpha}\left(x(1+t)^{\frac{1}{k}};k\right);$$
$$\forall\ m \in \{0,1,2,\cdots\},$$

where k is a positive integer ,

(2.3.3) $$\sum_{n=0}^{\infty} Y_n^{\alpha+\beta n}(x+ny;k)\, t^n = \frac{(1-\xi)^{-(\alpha+1)/k} \exp\left(x\left[1-(1-\xi)^{-\frac{1}{k}}\right]\right)}{1-k^{-1}\xi(1-\xi)^{-1}\left[\beta - y(1-\xi)^{-\frac{1}{k}}\right]},$$

where $\xi$ is a function of t defined by

(2.3.4) $$\xi = t(1-\xi)^{-\beta/k} \exp\left(y\left[1-(1-\xi)^{-\frac{1}{k}}\right]\right), \quad \xi(0)=0,$$

(2.3.5) $$\sum_{n=0}^{\infty} Y_n^{\alpha+\beta n}(x+ny;k)\, t^n = \frac{(1+\eta)^{(\alpha+1)/k} \exp\left(x\left[1-(1+\eta)^{\frac{1}{k}}\right]\right)}{1-k^{-1}\eta\left[\beta - y(1+\eta)^{\frac{1}{k}}\right]},$$

where $\eta$ is a function of t defined by

(2.3.6) $$\eta = t(1+\eta)^{\frac{(\beta+k)}{k}} \exp\left(y\left[1-(1+\eta)^{\frac{1}{k}}\right]\right), \quad \eta(0)=0.$$

To prove these results, we shall prove the following results for $G_n^{(\alpha)}(x,r,p,k)$:

$$(2.3.7)\quad \sum_{n=0}^{\infty} t^n G_n^{(\alpha-kn)}(x,r,p,k) = (1+kt)^{\frac{(\alpha-k)}{k}} \exp(px^r[1-(1+kt)^{r/k}]).$$

$$(2.3.8)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} G_{m+n}^{(\alpha-kn)}(x,r,p,k)\,t^n$$
$$= (1+kt)^{(\alpha-k)/k} \exp(px^r[1-(1+kt)^{r/k}]) \times G_m^{(\alpha)}(x(1+kt)^{\frac{1}{k}},r,p,k).$$

$$(2.3.9)\quad \sum_{n=0}^{\infty} G_n^{(\alpha+\beta n)}([x^r+ny^r]^{\frac{1}{r}},r,p,k]\,t^n$$
$$= \frac{(1-\upsilon)^{-\frac{\alpha}{k}} \exp(px^r[1-(1-\upsilon)^{-r/k}])}{1-k^{-1}\upsilon(1-\upsilon)^{-1}[\beta-rpy^r(1-\upsilon)^{-\frac{r}{k}}]}.$$

$$(2.3.10)\quad \sum_{n=0}^{\infty} G_n^{(\alpha+\beta n)}([x^r+ny^r]^{\frac{1}{r}},r,p,k)\,t^n$$
$$= \frac{(1+\upsilon)^{\alpha/k} \exp(px^r[1-(1+\upsilon)^{r/k}])}{1-k^{-1}\upsilon[\beta-rpy^r(1+\upsilon)^{r/k}]},$$

where $\xi$ and $\eta$ are defined by (2.3.4) and (2.3.6) respectively.

Now from equation (1.4.21), and letting $u = x^{-k}$, we have

$$\sum_{n=0}^{\infty} t^n G_n^{(\alpha-kn)}(x,r,p,k) = x^{-\alpha}\exp(px^r)\sum_{n=0}^{\infty}\frac{t^n}{n!}(x^{k+1}D_x)^n\left[x^{\alpha-kn}\exp(-px^r)\right]$$

$$= x^{-\alpha}\exp(px^r)\sum_{n=0}^{\infty}\frac{(-kt)^n}{n!}\frac{d^n}{du^n}\left[u^n.u^{-\frac{\alpha}{k}}.\exp(-pu^{-\frac{r}{k}})\right]$$

$$= x^{-\alpha}\exp(px^r)\left[(u(1+kt)^{-1})^{-\frac{\alpha}{k}}\exp(-p[u(1+kt)^{-1}]^{-\frac{r}{k}})\right].(1+kt)^{-1}$$

[ by Lagranges theorem]

$$= x^{-\alpha}\exp(px^r)\left[x^{\alpha}(1+kt)^{\frac{\alpha}{k}}\exp\{-px^r(1+kt)^{\frac{r}{k}}\}\right].(1+kt)^{-1}$$

$$= (1+kt)^{\frac{(\alpha-k)}{k}} \exp\left[px^r\left(1-(1+kt)^{\frac{r}{k}}\right]\right. .$$

This proves (2.3.7).

From (2.3.7), we have

$$\sum_{n=0}^{\infty}\left(\frac{t}{k}\right)^n G_n^{(\alpha-kn)}(x,r,p,k) = (1+t)^{(\alpha-k)/k} \exp\left(px^r\left[1-(1+t)^{\frac{r}{k}}\right]\right).$$

Hence for r = p = 1 and $\alpha$ is replaced by $\alpha+1$, and using (1.4.22), we get

$$\sum_{n=0}^{\infty} Y_n^{\alpha-kn}(x;k)\, t^n = (1+t)^{(\alpha-k+1)/k} \exp\left(x\left[1-(1+t)^{\frac{1}{k}}\right]\right),$$

which proves (2.3.1).

Again from equation (1.4.21), and letting $u = x^{-k}$, we have

$$\sum_{n=0}^{\infty}\binom{m+n}{n} G_{m+n}^{(\alpha-kn)}(x,r,p,k)\, t^n$$

$$= x^{-km-\alpha} e^{px^r}(x^{k+1}D_x)^m \sum_{n=0}^{\infty} \frac{(m+n)!\; t^n}{(m+n)!\; m!\, n!}(x^{k+1}D_x)^n\left\{x^{\alpha-kn} e^{-px^r}\right.$$

$$= \frac{x^{-km-\alpha} e^{px^r}(x^{k+1}D_x)^m}{m!} \sum_{n=0}^{\infty} \frac{(-kt)^n}{n!} \frac{d^n}{du^n}\left\{u^n u^{-\frac{\alpha}{k}} \exp(-pu^{-\frac{r}{k}}\right.$$

$$= \frac{x^{-km-\alpha} e^{px^r}(x^{k+1}D_x)^m}{m!}\left\{\left[u(1+kt)^{-1}\right]^{-\frac{\alpha}{k}} \exp\left[-p\left(u(1+kt)^{-1}\right]^{-\frac{r}{k}}\right]\right.$$

$$\cdot (1+kt)^{-1}$$

[ by Lagrange's theorem ]

$$= x^{-km-\alpha} e^{px^r} (1+kt)^{-m-1} x^{km+\alpha} (1+kt)^{m+\frac{\alpha}{k}} .$$

$$\cdot \exp\left(-px^r(1+kt)^{r/k}\right) \times G_m^{(\alpha)}\left(x(1+kt)^{\frac{1}{k}}, r, p, k\right)$$

$$= (1+kt)^{(\alpha-k)/k} \exp\left(px^r\left[1-(1+kt)^{\frac{r}{k}}\right]\right) \times G_m^{(\alpha)}\left(x(1+kt)^{\frac{1}{k}}, r, p, k\right.$$

This proves (2.3.8).

From (2.3.8), we have

$$\sum_{n=0}^{\infty}\binom{m+n}{n} G_{m+n}^{(\alpha-kn)}(x,r,p,k)\left(\frac{t}{k}\right)^n = (1+t)^{(\alpha-k)/k}\exp\left(px^r\left[1-(1+t)^{\frac{r}{k}}\right]\right)$$
$$\times\, G_m^{(\alpha)}\left(x(1+t)^{\frac{1}{k}}, r, p, k\right).$$

Hence for r = p = 1 and $\alpha$ is replaced by $\alpha+1$ and using (1.4.22), we get

$$\sum_{n=0}^{\infty}\binom{m+n}{n} Y_{m+n}^{\alpha-kn}(x;k)t^n = (1+t)^{(\alpha-k+1)/k}\exp\left(x\left[1-(1+t)^{\frac{1}{k}}\right]\right)$$
$$\times\, Y_m^{\alpha}\left(x(1+t)^{\frac{1}{k}}; k\right),$$

which proves (2.3.2).

Now from (2.2.3), replacing x by $x^{\frac{1}{r}}$, we get

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}\left(x^{\frac{1}{r}}, r, p, k\right) z^n = (1-kz)^{-\alpha/k}\exp\left[px\left(1-(1-kz)^{-\frac{r}{k}}\right)\right].$$

Now by use of the extended Carlitz theorem ( see Appendix ), we get

$$\sum_{n=0}^{\infty} G_n^{(\alpha+\beta n)}\left[(x+ny)^{\frac{1}{r}}, r, p, k\right]t^n = \frac{(1-u)^{-\frac{\alpha}{k}}\exp\left(px\left[1-(1-u)^{-\frac{r}{k}}\right]\right)}{1-u\left\{\beta k^{-1}(1-u)^{-1} - rpyk^{-1}(1-u)^{-\frac{r}{k}-1}\right\}}$$

$$= \frac{(1-u)^{-\frac{\alpha}{k}}\exp\left(px\left[1-(1-u)^{-\frac{r}{k}}\right]\right)}{1-uk^{-1}(1-u)^{-1}\left[\beta - rpy(1-u)^{-\frac{r}{k}}\right]}, k \neq 0,$$

where $u = kt(1-u)^{-\beta/k}\exp\left(py\left[1-(1-u)^{-r/k}\right]\right)$.

Now substituting $x^r$ for x and $y^r$ for y, we get,

$$\sum_{n=0}^{\infty} G_n^{(\alpha+\beta n)}\left[(x^r+ny^r)^{\frac{1}{r}}, r, p, k\right]t^n = \frac{(1-u)^{-\alpha/k}\exp\left(px^r\left[1-(1-u)^{-\frac{r}{k}}\right]\right)}{1-uk^{-1}(1-u)^{-1}\left[\beta - rpy^r(1-u)^{-\frac{r}{k}}\right]},$$

where

$$u = kt(1-u)^{-\beta/k}\exp\left(py^r\left[1-(1-u)^{-\frac{r}{k}}\right]\right).$$

This proves (2.3.9).

Now for r = p = 1 and $\alpha$ is replaced by $\alpha+1$, and using (1.4.22), we get

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\beta n}(x+ny;k)\,t^n = \frac{(1-\xi)^{-(\alpha+1)/k}\exp\left(x\left[1-(1-\xi)^{-\frac{1}{k}}\right]\right)}{1-k^{-1}\xi(1-\xi)^{-1}\left[\beta-y(1-\xi)^{-\frac{1}{k}}\right]},$$

$$\xi = t(1-\xi)^{-\beta/k}\exp\left(y\left[1-(1-\xi)^{-\frac{1}{k}}\right]\right).$$

This proves (2.3.3).

Now substituting $u = \frac{v}{1+v}$ in equation (2.3.9)

$$\sum_{n=0}^{\infty} G_n^{(\alpha+\beta n)}\left([x^r+ny^r]^{\frac{1}{r}}, r, p, k\right) = \frac{\left(1-\frac{v}{1+v}\right)^{-\frac{\alpha}{k}}\exp\left(px^r\left[1-\left(1-\frac{v}{1+v}\right)^{-\frac{r}{k}}\right]\right)}{1-k^{-1}\cdot\frac{v}{1+v}\left(1-\frac{v}{1+v}\right)^{-1}\left[\beta-rpy^r\left(1-\frac{v}{1+v}\right)^{-\frac{r}{k}}\right]}$$

$$= \frac{(1+v)^{\alpha/k}\exp\left(px^r\left[1-(1+v)^{\frac{r}{k}}\right]\right)}{1-k^{-1}v\left[\beta-rpy^r(1+v)^{\frac{r}{k}}\right]},$$

where $v = kt(1+v)^{(\beta+k)/k}\exp\left(py^r\left[1-(1+v)^{\frac{r}{k}}\right]\right)$.

This proves (2.3.10).

Now for r = p = 1 and $\alpha$ is replaced by $\alpha+1$, and using (1.4.22), we get

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\beta n}(x+ny;k)\,t^n = \frac{(1+\eta)^{\frac{(\alpha+1)}{k}}\exp\left(x\left[1-(1+\eta)^{\frac{1}{k}}\right]\right)}{1-k^{-1}\eta\left[\beta-y(1+\eta)^{\frac{1}{k}}\right]},$$

$$\eta = t(1+\eta)^{(\beta+k)/k}\exp\left(y\left[1-(1+\eta)^{\frac{1}{k}}\right]\right).$$

This proves (2.3.5).

## 2.4 RECURRENCE RELATIONS & OPERATIONAL FORMULAS:

$Y_n^{\alpha}(x;k)$ satisfies the following recurrence and operational formulas:

(2.4.1) $k(n+1)\,Y_{n+1}^{\alpha}(x;k) = x D_x Y_n^{\alpha}(x;k) + (kn+\alpha-x+1)\,Y_n^{\alpha}(x;k)$

(2.4.2) $D_x Y_n^{\alpha}(x;k) = Y_n^{\alpha}(x;k) - Y_n^{\alpha+1}(x;k)$,

(2.4.3) $(\alpha-k+1)\, Y_n^{\alpha}(x;k) = x\, Y_n^{\alpha+1}(x;k) + (n+1)k\, Y_{n+1}^{\alpha-k}(x;k)$ ,

(2.4.4) $k(n+1)\, Y_{n+1}^{\alpha}(x;k) = (kn+\alpha+1)\, Y_n^{\alpha}(x;k) - x\, Y_n^{\alpha+1}(x;k)$ ,

(2.4.5) $Y_{n+1}^{\alpha-k}(x;k) = Y_{n+1}^{\alpha}(x;k) - Y_n^{\alpha}(x;k)$ ,

(2.4.6) $\prod_{j=0}^{n-1}(\delta+\alpha+kj-x+1) = k^n n! \sum_{j=0}^{n} \frac{(kx^k)^{-j}}{j!}\, Y_{n-j}^{\alpha}(x;k)(x^{k+1}D_x)^j$ ,

(2.4.7) $Y_n^{\alpha}(x;k) = \frac{1}{k^n n!} \prod_{j=0}^{n-1}(\delta+\alpha+jk-x+1)\cdot 1$ ;

where $\delta = xD_x$ ,

(2.4.8) $D_x^m\{e^{-x}Y_n^{\alpha}(x;k)\} = (-1)^m e^{-x}\, Y_n^{\alpha+m}(x;k)$ .

Proof of (2.4.1)

From equation (2.1.2)

$$Y_n^{\alpha}(x;k) = \frac{x^{-kn-\alpha-1}}{k^n n!} e^x (x^{k+1}D_x)^n \{x^{\alpha+1}e^{-x}\} .$$

consider

$$xD_x Y_n^{\alpha}(x;k) = \frac{1}{k^n n!}\Big[(-kn-\alpha-1)\, x^{-kn-\alpha-1} e^x (x^{k+1}D_x)^n \{x^{\alpha+1}e^{-x}\}$$
$$+ x^{-kn-\alpha} e^x (x^{k+1}D_x)^n\{x^{\alpha+1}e^{-x}\} + x^{-kn-\alpha-1-k}\cdot e^x \cdot$$
$$(x^{k+1}D_x)(x^{k+1}D_x)^{n+1}\{x^{\alpha+1}e^{-x}\}\Big]$$

$$= \frac{1}{k^n n!}\Big[(-kn-\alpha-1)\, k^n n!\, Y_n^{\alpha}(x;k) + x k^n n!\, Y_n^{\alpha}(x;k)$$
$$+ k^{n+1}(n+1)!\, Y_{n+1}^{\alpha}(x;k)\Big]$$

$$= -(kn+\alpha+1)\,Y_n^{\alpha}(x;k) + x\,Y_n^{\alpha}(x;k) + k(n+1)\,Y_{n+1}^{\alpha}(x;k)$$

Hence, on transposition, we get

$$k(n+1)\,Y_{n+1}^{\alpha}(x;k) = x\,D_x Y_n^{\alpha}(x;k) + (kn+\alpha-x+1)\,Y_n^{\alpha}(x;k)$$

which proves (2.4.1).

Proof of (2.4.2):

Differenciating relation (2.2.1) with respect to x, we get

$$D_x \sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\,t^n = (1-t)^{-(\alpha+1)/k}\left[1-(1-t)^{-\frac{1}{k}}\right] e^{(x[1-(1-t)^{-\frac{1}{k}}])}$$

$$= (1-t)^{-(\alpha+1)/k}\, e^{x[1-(1-t)^{-\frac{1}{k}}]} - (1-t)^{-\left[\frac{(\alpha+1)+1}{k}\right]} e^{(x[1-(1-t)^{-\frac{1}{k}}])}$$

$$= \sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\,t^n - \sum_{n=0}^{\infty} Y_n^{\alpha+1}(x;k)\,t^n .$$

Hence comparing the coefficients of $t^n$ both sides, we get

$$D_x Y_n^{\alpha}(x;k) = Y_n^{\alpha}(x;k) - Y_n^{\alpha+1}(x;k) ,$$

which proves (2.4.2).

Proof of (2.4.3):

Differenciating relation (2.2.1) with respect to t, we get

$$\sum_{n=1}^{\infty} Y_n^{\alpha}(x;k)\,n\cdot t^{n-1} = -\left(\frac{\alpha+1}{k}\right)(1-t)^{-\left(\frac{\alpha+1}{k}\right)-1}(-1)\, e^{(x[1-(1-t)^{-\frac{1}{k}}])}$$

$$+ (1-t)^{-\left(\frac{\alpha+1}{k}\right)} \exp\left(x\left[1-(1-t)^{-\frac{1}{k}}\right]\right)\cdot \left(x\left(-\tfrac{1}{k}\right)(1-t)^{-\frac{1}{k}-1}\right)$$

$$= \left(\frac{\alpha+1}{k}\right)\sum_{n=0}^{\infty} Y_n^{\alpha+k}(x;k)\,t^n - \frac{x}{k}\sum_{n=0}^{\infty} Y_n^{\alpha+k+1}(x;k)\,t^n$$

or

$$k(n+1)\sum_{n=0}^{\infty} Y_{n+1}^{\alpha}(x;k)\,t^n = (\alpha+1)\sum_{n=0}^{\infty} Y_n^{\alpha+k}(x;k)\,t^n - x\sum_{n=0}^{\infty} Y_n^{\alpha+k+1}(x;k)\,t^n$$

On comparing the coefficients of $t^n$ on both sides we get

$$k(n+1)\,Y_{n+1}^{\alpha}(x;k) = (\alpha+1)\,Y_n^{\alpha+k}(x;k) - x\,Y_n^{\alpha+k+1}(x;k).$$

Replacing $\alpha$ by $\alpha-k$, we get

$$k(n+1)\,Y_{n+1}^{\alpha-k}(x;k) = (\alpha-k+1)\,Y_n^{\alpha}(x;k) - x\,Y_n^{\alpha+1}(x;k)$$

or

$$(\alpha-k+1)\,Y_n^{\alpha}(x;k) = x\,Y_n^{\alpha+1}(x;k) + k(n+1)\,Y_{n+1}^{\alpha-k}(x;k),$$

which proves (2.4.3).

Proof of (2.4.4):

Consider

$$xD_x\left(x^{\alpha+kn+1}\,Y_n^{\alpha}(x;k)\right) = (\alpha+kn+1)\,x^{kn+\alpha+1}\,Y_n^{\alpha}(x;k) + x^{kn+\alpha+1}\cdot xD_x Y_n^{\alpha}(x;k)$$

or

$$xD_x\left[\frac{e^x}{k^n n!}\,(x^{k+1}D_x)^n\{x^{\alpha+1}e^{-x}\}\right] = (\alpha+kn+1)\,x^{kn+\alpha+1}\,Y_n^{\alpha}(x;k) + x^{kn+\alpha+1}\cdot xD_x Y_n^{\alpha}(x;k)$$

or

$$\frac{x\cdot e^x}{k^n n!}\,(x^{k+1}D_x)^n\{x^{\alpha+1}e^{-x}\} + \frac{e^x x^{-k}}{k^n n!}\,(x^{k+1}D_x)(x^{k+1}D_x)^n\{x^{\alpha+1}e^{-x}\}$$
$$= (\alpha+kn+1)\,x^{kn+\alpha+1}\,Y_n^{\alpha}(x;k) + x^{kn+\alpha+1}\cdot xD_x Y_n^{\alpha}(x;k)$$

or

$$x\cdot x^{kn+\alpha+1}\,Y_n^{\alpha}(x;k) + k(n+1)\,x^{kn+\alpha+1}\,Y_{n+1}^{\alpha}(x;k)$$
$$= (\alpha+kn+1)\,x^{kn+\alpha+1}\,Y_n^{\alpha}(x;k) + x^{kn+\alpha+1}\cdot x\cdot D_x Y_n^{\alpha}(x;k)$$

or

$$x\,Y_n^{\alpha}(x;k) + k(n+1)\,Y_{n+1}^{\alpha}(x;k) = (\alpha+kn+1)\,Y_n^{\alpha}(x;k) + xD_x Y_n^{\alpha}(x;k)$$

or $x Y_n^{\alpha}(x;k) + k(n+1) Y_{n+1}^{\alpha}(x;k) = (\alpha+kn+1) Y_n^{\alpha}(x;k)$
$$+ x Y_n^{\alpha}(x;k) - x Y_n^{\alpha+1}(x;k)$$

[by use of (2.4.2)]

or $k(n+1) Y_{n+1}^{\alpha}(x;k) = (kn+\alpha+1) Y_n^{\alpha}(x;k) - x Y_n^{\alpha+1}(x;k)$,

which proves (2.4.4).

Proof of (2.4.5):

Eliminating $x Y_n^{\alpha+1}(x;k)$ between (2.4.3) and (2.4.4), by adding we get

$$(\alpha-k+1) Y_n^{\alpha}(x;k) + k(n+1) Y_{n+1}^{\alpha}(x;k) = k(n+1) Y_{n+1}^{\alpha-k}(x;k) + (kn+\alpha+1) Y_n^{\alpha}(x;k).$$

Or $k(n+1) Y_{n+1}^{\alpha-k}(x;k) = k(n+1) Y_{n+1}^{\alpha}(x;k) + (\alpha-k+1-kn-\alpha-1) Y_n^{\alpha}(x;k)$

Or $k(n+1) Y_{n+1}^{\alpha-k}(x;k) = k(n+1) Y_{n+1}^{\alpha}(x;k) - k(n+1) Y_n^{\alpha}(x;k)$

Or $Y_{n+1}^{\alpha-k}(x;k) = Y_{n+1}^{\alpha}(x;k) - Y_n^{\alpha}(x;k)$,

which proves (2.4.5).

Proof of (2.4.6):

Since

$$(x^{k+1}D_x)^n \left[\{x^{\alpha+1} e^{-x}\} f\right] = (x^{k+1}D_x)^{n-1} (x^{k+1}D_x) \left[\{x^{\alpha+1} e^{-x}\} f\right]$$
$$= (x^{k+1}D_x)^{n-1} x^{k+1} \left\{ (\alpha+1) x^{\alpha} e^{-x} f - x^{\alpha+1} e^{-x} f + x^{\alpha+1} e^{-x} D_x f \right\}$$
$$= (x^{k+1}D_x)^{n-1} \left\{ (\alpha+1) x^{\alpha+k+1} e^{-x} f - x^{\alpha+k+2} e^{-x} \cdot f + x^{\alpha+k+2} e^{-x} D_x f \right\}$$

$$= (x^{k+1}D_x)^{n-1}\left[x^{\alpha+k+1}e^{-x}(xD_x - x + \alpha+1)f\right]$$

$$= (x^{k+1}D_x)^{n-2}\, x^{k+1}\left[(\alpha+k+1)x^{\alpha+k}e^{-x} - x^{\alpha+k+1}e^{-x} + x^{\alpha+k+1}e^{-x}D_x\right](xD_x - x+\alpha+1)f$$

$$= (x^{k+1}D_x)^{n-2}\left[x^{\alpha+2k+1}e^{-x}\{(\alpha+k+1) - x + xD_x\}\right]\times (xD_x - x+\alpha+1)f ,$$

and repeating this process upto n times, we get

$$= x^{\alpha+kn+1}e^{-x}\prod_{j=0}^{n-1}(xD_x - x + \alpha + kj+1)f .$$

Thus

$$(2.4.9)\quad (x^{k+1}D)^n\left[\{x^{\alpha+1}e^{-x}\}f\right] = x^{\alpha+kn+1}e^{-x}\prod_{j=0}^{n-1}(\delta - x+\alpha+kj+1)f$$

Also, we have, using (1.5.3)

$$(x^{k+1}D_x)^n\left[\{x^{\alpha+1}e^{-x}\}f\right] = \sum_{j=0}^{n}\binom{n}{j}(x^{k+1}D_x)^{n-j}(x^{\alpha+1}e^{-x})(x^{k+1}D_x)^j f$$

$$= \sum_{j=0}^{n}\binom{n}{j}Y_{n-j}^{\alpha}(x;k)\,k^{n-j}(n-j)!\,x^{k(n-j)+\alpha+1} \cdot e^{-x}(x^{k+1}D_x)^j f ,$$

i.e.

$$(2.4.10)\quad (x^{k+1}D_x)^n\left[\{x^{\alpha+1}e^{-x}\}f\right] = k^n n!\sum_{j=0}^{n}\frac{(kx^k)^{-j}}{j!}Y_{n-j}^{\alpha}(x;k)\,x^{kn+\alpha+1} \cdot e^{-x}(x^{k+1}D_x)^j f .$$

Hence equating (2.4.9) and (2.4.10), we get

$$k^n n!\sum_{j=0}^{n}\frac{(kx^k)^{-j}}{j!}Y_{n-j}^{\alpha}(x;k)\,x^{kn+\alpha+1}e^{-x}(x^{k+1}D_x)^j f$$

$$= x^{\alpha+kn+1}e^{-x}\prod_{j=0}^{n-1}(\delta - x+\alpha+kj+1)f .$$

or $\displaystyle k^n n! \sum_{j=0}^{n} \frac{(kx^k)^{-j}}{j!} Y_{n-j}^{\alpha}(x;k)(x^{k+1}D_x)^j = \prod_{j=0}^{n-1}(\delta - x + \alpha + kj + 1)$

or $\displaystyle \prod_{j=0}^{n-1}(\delta - x + \alpha + kj + 1) = k^n n! \sum_{j=0}^{n} \frac{(kx^k)^{-j}}{j!} Y_{n-j}^{\alpha}(x;k)(x^{k+1}D_x)^j$

which proves (2.4.6).

Proof of (2.4.7):

From (2.4.9)

$$(x^{k+1}D_x)^n\left[\{x^{\alpha+1}e^{-x}\}f\right] = x^{\alpha+kn+1}e^{-x}\prod_{j=0}^{n-1}(\delta - x + \alpha + kj + 1)\cdot f\,.$$

Now when f = 1, we get

$$(x^{k+1}D_x)^n\left[\{x^{\alpha+1}e^{-x}\}\right] = x^{\alpha+kn+1}e^{-x}\prod_{j=0}^{n-1}(\delta - x + \alpha + kj + 1)\cdot 1$$

$$Y_n^{\alpha}(x;k)\, k^n n!\, x^{kn+1+\alpha}e^{-x} = x^{\alpha+kn+1}e^{-x}\prod_{j=0}^{n-1}(\delta - x + \alpha + kj + 1)\cdot 1\,.$$

Hence

$$Y_n^{\alpha}(x;k) = \frac{1}{k^n n!}\prod_{j=0}^{n-1}(\delta + \alpha + kj - x + 1)\cdot 1$$

which proves (2.4.7).

To prove (2.4.8) we first prove the following relation for $G_n^{(\alpha)}(x,\gamma,p,k)$:

$$(2.4.11)\quad (x^{1-\gamma}D_x)^m\{\exp(-px^{\gamma})\}\, G_n^{(\alpha)}(x,\gamma,p,k)$$
$$= (-\gamma p)^m \exp(-px^{\gamma})\, G_n^{(\alpha+m\gamma)}(x,\gamma,p,k)\,.$$

Differenciating relation (2.2.3) with respect to x, we get

$$\sum_{n=0}^{\infty} D_x G_n^{(\alpha)}(x,\gamma,p,k)\, t^n = (1-kt)^{-\frac{\alpha}{k}} \exp\left[px^{\gamma}\left(1-(1-kt)^{-\frac{\gamma}{k}}\right)\right]$$
$$\cdot\, p\gamma x^{\gamma-1}\left[1-(1-kt)^{-\frac{\gamma}{k}}\right]$$

$$= pr x^{r-1}(1-kt)^{-\alpha/k}\exp\left(px^r\left[1-(1-kt)^{-r/k}\right]\right)$$

$$- pr x^{r-1}(1-kt)^{-\frac{(\alpha+r)}{k}}\exp\left(px^r\left[1-(1-kt)^{-r/k}\right]\right),$$

or $\sum_{n=0}^{\infty} D_x G_n^{(\alpha)}(x,r,p,k)t^n$

$$= prx^{r-1}\sum_{n=0}^{\infty} t^n G_n^{(\alpha)}(x,r,p,k) - prx^{r-1}\sum_{n=0}^{\infty} t^n G_n^{(\alpha+r)}(x,r,p,k).$$

Equating the coefficients of $t^n$ both sides,

$$D_x G_n^{(\alpha)}(x,r,p,k) = prx^{r-1}G_n^{(\alpha)}(x,r,p,k) - prx^{r-1}G_n^{(\alpha+r)}(x,r,p,k)$$

or

$$(2.4.12)\quad (D_x - prx^{r-1})\, G_n^{(\alpha)}(x,r,p,k) = -prx^{r-1}G_n^{(\alpha+r)}(x,r,p,k)$$

Using the shift formula (1.5.6), we get

$$\exp(px^r)\left[D_x\{\exp(-px^r)\, G_n^{(\alpha)}(x,r,p,k)\}\right]$$

$$= -prx^{r-1}G_n^{(\alpha+r)}(x,r,p,k)$$

Or $D_x\left[\exp(-px^r)\, G_n^{(\alpha)}(x,r,p,k)\right]$

$$= -prx^{r-1}\exp(-px^r)\, G_n^{(\alpha+r)}(x,r,p,k)$$

Or

$$(2.4.13)\quad (x^{1-r}D_x)\left[\exp(-px^r)\, G_n^{(\alpha)}(x,r,p,k)\right]$$

$$= -pr\exp(-px^r)\, G_n^{(\alpha+r)}(x,r,p,k).$$

Now repeating operation of $(x^{1-r}D)$, m times, we get

$$(x^{1-r}D_x)^m\left[\exp(-px^r)\, G_n^{(\alpha)}(x,r,p,k)\right] = (-pr)^m\exp(-px^r)\, G_n^{(\alpha+mr)}(x,r,p,k).$$

This proves (2.4.11).

Hence for r = p = 1 and $\alpha$ is replaced by $\alpha+1$, and using (1.4.22), we get

$$D_x^m \{ e^{-x} Y_n^{\alpha}(x;k) \} = (-1)^m e^{-x} Y_n^{\alpha+m}(x;k) ,$$

which proves (2.4.8).

As consequence of (2.4.8), we have the following generating relation:

$$(2.4.14) \quad \sum_{n=0}^{\infty} Y_m^{\alpha+n}(x;k) \frac{t^n}{n!} = e^{t} Y_m^{\alpha}(x-t;k).$$

To prove (2.4.14), we consider

$$\sum_{n=0}^{\infty} Y_m^{\alpha+n}(x;k) \frac{t^n}{n!} = \sum_{n=0}^{\infty} e^{x} (-1)^n \frac{t^n}{n!} D^n \{ e^{-x} Y_m^{\alpha}(x;k) \}$$

$$= e^{x} . \exp(-(x-t)) Y_m^{\alpha}(x-t;k)$$

[by use of (1.5.2)]

$$= \exp(t) \cdot Y_m^{\alpha}(x-t;k) ,$$

which proves (2.4.14).

By similar method, we can prove analogous result for $G_m^{(\alpha)}(x,r,p,k)$ as

$$(2.4.15) \quad \sum_{n=0}^{\infty} G_m^{(\alpha+nr)}(x,r,p,k) \frac{t^n}{n!} = e^{t} G_m^{(\alpha)}\left( [x^r - t/p]^{1/r}; r, p, k \right).$$

## 2.5 FINITE SUMS:

Following are the finite sums for $Y_n^{\alpha}(x;k)$:

$$(2.5.1) \quad Y_n^{\alpha}(x;k) = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} Y_{n-j}^{\alpha-k+kn}(x;k) ,$$

$$(2.5.2) \quad Y_n^{\alpha}(x;k) = \sum_{j=0}^{n} \binom{(\alpha-\beta)/k}{j} Y_{n-j}^{\beta+kj}(x;k) ,$$

$$(2.5.3)\quad Y_n^{\alpha}(x;k) = \sum_{j=0}^{n-1} \binom{n-1}{j} Y_{n-j}^{\alpha+k-kn+kj}(x;k),$$

$$(2.5.4)\quad Y_n^{\alpha}(x;k) = \sum_{j=0}^{n} \binom{j-1+(\alpha-\beta)/k}{j} Y_{n-j}^{\beta}(x;k),$$

$$(2.5.5)\quad Y_n^{\alpha+\beta+1}(x+y;k) = \sum_{j=0}^{n} Y_j^{\alpha}(x;k)\, Y_{n-j}^{\beta}(y;k).$$

To prove all these relations we first prove the following finite sums for $G_n^{(\alpha)}(x,r,p,k)$:

$$(2.5.6)\quad G_n^{(\alpha)}(x,r,p,k) = \sum_{j=0}^{n-1} (-k)^j \binom{n-1}{j} G_{n-j}^{(\alpha-k+kn)}(x,r,p,k),$$

$$(2.5.7)\quad G_n^{(\alpha)}(x,r,p,k) = \sum_{j=0}^{n} k^j \binom{(\alpha-\beta)/k}{j} G_{n-j}^{(\beta+kj)}(x,r,p,k),$$

$$(2.5.8)\quad G_n^{(\alpha)}(x,r,p,k) = \sum_{j=0}^{n-1} k^j \binom{n-1}{j} G_{n-j}^{(\alpha+k-kn+kj)}(x,r,p,k),$$

$$(2.5.9)\quad G_n^{(\alpha)}(x,r,p,k) = \sum_{j=0}^{n} k^j \binom{j-1+(\alpha-\beta)/k}{j} G_{n-j}^{(\beta)}(x,r,p,k),$$

$$(2.5.10)\quad G_n^{(\alpha+\beta)}([x^r+y^r]^{1/r},r,p,k) = \sum_{j=0}^{n} G_j^{(\alpha)}(x,r,p,k)\, G_{n-j}^{(\beta)}(y,r,p,k).$$

From the equation (1.4.21), we have

$$G_n^{(\alpha)}(x,r,p,k) = \frac{x^{-kn-\alpha}\exp(px^r)}{n!}(x^{k+1}D_x)^n\left[x^{\alpha}\exp(-px^r)\right]$$

$$= \frac{x^{-kn-\alpha}\exp(px^{r})}{n!}(x^{k+1}D_x)^n\left[x^{\alpha-k+kn}\exp(-px^{r})\,x^{-k(n-1)}\right]$$

$$= \frac{x^{-kn-\alpha}\exp(px^{r})}{n!}\sum_{j=0}^{n}\binom{n}{j}\theta^{j}\left(x^{-k(n-1)}\right)\theta^{n-j}\left(x^{\alpha-k+kn}\exp(-px^{r})\right)$$

[by use of relation (1.5.3)]

$$= \sum_{j=0}^{n}\frac{1}{n!}\frac{n!\,(n-j)!}{j!\,(n-j)!}\left(-k(n-1)\right)^{(k,j)}G_{n-j}^{(\alpha-k+kn)}(x,r,p,k)$$

$$= \sum_{j=0}^{n-1}\binom{n-1}{j}(-k)^{j}\,G_{n-j}^{(\alpha-k+kn)}(x,r,p,k).$$

This proves (2.5.6).

Hence for p = r = 1 and $\alpha$ is replaced by $\alpha+1$, and using (1.4.22), we get

$$Y_n^{\alpha}(x;k) = \sum_{j=0}^{n-1}(-1)^{j}\binom{n-1}{j}Y_{n-j}^{\alpha-k+kn}(x;k),$$

which proves (2.5.1).

Again from relation (2.2.3)

$$\sum_{n=0}^{\infty}G_n^{(\alpha)}(x,r,p,k)\,t^{n} = (1-kt)^{-\frac{\alpha}{k}}\exp\left(px^{r}\left[1-(1-kt)^{-\frac{r}{k}}\right]\right)$$

$$= (1-kt)^{\frac{\beta-\alpha}{k}}(1-kt)^{-\frac{\beta}{k}}\exp\left(px^{r}\left[1-(1-kt)^{-\frac{r}{k}}\right]\right)$$

$$= \left[1+\frac{kt}{1-kt}\right]^{\frac{\alpha-\beta}{k}}(1-kt)^{-\frac{\beta}{k}}\exp\left(px^{r}\left[1-(1-kt)^{-\frac{r}{k}}\right]\right)$$

$$= \sum_{j=0}^{\infty}\left(\frac{\beta-\alpha}{k}\right)_j(-1)^{j}(kt)^{j}(1-kt)^{-\frac{(\beta+kj)}{k}}\exp\left(px^{r}\left[1-(1-kt)^{-\frac{r}{k}}\right]\right)$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{\beta-\alpha}{k}\right)_j(-1)^{j}k^{j}\,G_n^{(\beta+kj)}(x,r,p,k)\,t^{n+j}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \left(\frac{\beta-\alpha}{k}\right)_j (-1)^j k^j G_{n-j}^{(\beta+kj)}(x,r,p,k)\, t^n .$$

On comparing the coefficients of $t^n$ both sides, we get

$$G_n^{(\alpha)}(x,r,p,k) = \sum_{j=0}^{n} k^j \binom{(\alpha-\beta)/k}{j} G_{n-j}^{(\beta+kj)}(x,r,p,k) .$$

This proves (2.5.7).

Hence for r = p = 1 and $\alpha$ and $\beta$ are replaced by $\alpha+1$

and $\beta+1$ , and using (1.4.22), we get

$$Y_n^{\alpha}(x;k) = \sum_{j=0}^{n} \binom{(\alpha-\beta)/k}{j} Y_{n-j}^{\beta+kj}(x;k) ,$$

which proves (2.5.2).

Now substituting $\beta = \alpha + k - kn$ in equation (2.5.7), we have

$$G_n^{(\alpha)}(x,r,p,k) = \sum_{j=0}^{n-1} k^j \binom{n-1}{j} G_{n-j}^{(\alpha+k-kn+kj)}(x,r,p,k) .$$

This proves (2.5.8).

Hence for r = p = 1 and $\alpha$ is replaced by $\alpha+1$ , we get,

using (1.4.22)

$$Y_n^{\alpha}(x;k) = \sum_{j=0}^{n-1} \binom{n-1}{j} Y_{n-j}^{(\alpha+k-kn+kj)}(x;k) ,$$

which proves (2.5.3).

Again from (2.2.3)

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x,r,p,k)\, t^n = (1-kt)^{-\frac{\alpha}{k}} \exp\left(px^r\left[1-(1-kt)^{-\frac{r}{k}}\right]\right)$$

$$= (1-kt)^{\frac{\beta-\alpha}{k}} (1-kt)^{-\frac{\beta}{k}} \exp\left(px^r\left[1-(1-kt)^{-\frac{r}{k}}\right]\right)$$

$$= (1-kt)^{-\frac{(\alpha-\beta)}{k}} \sum_{n=0}^{\infty} G_n^{(\beta)}(x,r,p,k)\, t^n$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{((\alpha-\beta)/k)_j\, k^j}{j!} G_n^{(\beta)}(x,r,p,k)\, t^{n+j}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{((\alpha-\beta)/k)_j \, k^j}{j!} G_{n-j}^{(\beta)}(x,r,p,k)\, t^n .$$

On comparing the coefficient of $t^n$, on both sides, we get

$$G_n^{(\alpha)}(x,r,p,k) = \sum_{j=0}^{n} k^j \binom{j-1+(\alpha-\beta)/k}{j} G_{n-j}^{(\beta)}(x,r,p,k) .$$

This proves (2.5.9).

Hence for r = p = 1 and $\alpha$ and $\beta$ are replaced by $\alpha+1$ and $\beta+1$ respectively and using (1.4.22), we get

$$Y_n^{\alpha}(x;k) = \sum_{j=0}^{n} \binom{j-1+(\alpha-\beta)/k}{j} Y_{n-j}^{\beta}(x;k) ,$$

which proves (2.5.4).

Now from (2.2.3)

$$\sum_{n=0}^{\infty} G_n^{(\alpha+\beta)}[(x^r+y^r)^{\frac{1}{r}}, r, p, k]\, t^n = (1-kt)^{-\frac{(\alpha+\beta)}{k}} \exp\left(p(x^r+y^r)[1-(1-kt)^{-\frac{r}{k}}]\right)$$

$$= \sum_{n=0}^{\infty} G_n^{(\alpha)}(x,r,p,k)\, t^n \sum_{j=0}^{\infty} G_j^{(\beta)}(y,r,p,k)\, t^j$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} G_{n-j}^{(\alpha)}(x,r,p,k)\, G_j^{(\beta)}(y,r,p,k)\, t^n .$$

On comparing the coefficient of $t^n$, we get

$$G_n^{(\alpha+\beta)}([x^r+y^r]^{\frac{1}{r}}, r, p, k) = \sum_{j=0}^{n} G_{n-j}^{(\alpha)}(x,r,p,k)\, G_j^{(\beta)}(y,r,p,k)$$

or $$G_n^{(\alpha+\beta)}([x^r+y^r]^{\frac{1}{r}}, r, p, k) = \sum_{j=0}^{n} G_j^{(\alpha)}(x,r,p,k)\, G_{n-j}^{(\beta)}(y,r,p,k) .$$

This proves (2.5.10).

Hence for p = r = 1 and $\alpha$ and $\beta$ are replaced by $\alpha+1$ and $\beta+1$ respectively and using (1.4.22), we get

$$Y_n^{\alpha+\beta+1}(x+y;k) = \sum_{j=0}^{n} Y_j^{\alpha}(x;k)\, Y_{n-j}^{\beta}(y;k) \;,$$

which proves (2.5.5).

2.6 SOME MIXED MULTILATERAL GENERATING FUNCTIONS:

Following are the mixed multilateral generating functions involving $Y_n^{\alpha}(x;k)$:

(2.6.1) $$\sum_{n=0}^{\infty} Y_{m+n}^{\alpha}(x;k)\,\Lambda_n(y_1,\text{-------},y_N;z)t^n$$

$$= (1-t)^{-(\alpha+1)/k} \exp\left(x\left[1-(1-t)^{-\frac{1}{k}}\right]\right)$$

$$\times F\left[x(1-t)^{-\frac{1}{k}};y_1,\text{-----},y_N; zt^q/(1-t)^q\right]$$

(2.6.2) $$\sum_{n=0}^{\infty} Y_{m+n}^{\alpha-kn}(x;k)\,\Lambda_n(y_1,\text{-----},y_N;z)t^n$$

$$= (1+t)^{(\alpha-k+1)/k} \exp\left(x\left[1-(1+t)^{\frac{1}{k}}\right]\right)$$

$$\times G\left[x(1+t)^{\frac{1}{k}};y_1,\text{-----},y_N; zt^q/(1+t)^q\right],$$

where

(2.6.3) $$F[x;y_1,\text{-----},y_N;z] = \sum_{n=0}^{\infty} C_n\, Y_{m+qn}^{\alpha}(x;k)\,\Delta_n(y_1,\cdots,y_N)z^n,$$

(2.6.4) $$G[x;y_1,\text{-----},y_N;z] = \sum_{n=0}^{\infty} C_n\, Y_{m+qn}^{\alpha-kqn}(x;k)\,\Delta_n(y_1,\cdots,y_N)z^n,$$

(2.6.5) $$\Lambda_n(y_1,\text{-----},y_N;z)$$

$$= \sum_{j=0}^{[n/q]} \binom{m+n}{n-qj} c_j\, \Delta_j(y_1,\text{-----},y_N)\,z^j,$$

Proof of (2.6.1):

Consider

$$\sum_{n=0}^{\infty} Y_{m+n}^{\alpha}(x;k)\,\Lambda_n(y_1,\dots,y_N;z)\,t^n$$

$$= \sum_{n=0}^{\infty} Y_{m+n}^{\alpha}(x;k) \sum_{j=0}^{[n/q]} \binom{m+n}{n-qj} c_j \Delta_j(y_1,\dots,y_N) z^j t^n$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} Y_{m+n+qj}^{\alpha}(x;k)\binom{m+n+qj}{n} c_j \Delta_j(y_1,\dots,y_N) z^j t^{n+qj}$$

$$= \left(\sum_{j=0}^{\infty} t^{qj} c_j \Delta_j(y_1,\dots,y_N) z^j\right)\left(\sum_{n=0}^{\infty}\binom{m+n+qj}{n} Y_{m+n+qj}^{\alpha}(x;k)\, t^n\right)$$

$$= \sum_{j=0}^{\infty} t^{qj} c_j \Delta_j(y_1,\dots,y_N) z^j (1-t)^{-qj-(\alpha+1)/k} \exp\left(x\left[1-(1-t)^{-\frac{1}{k}}\right]\right) \times Y_{m+qj}^{\alpha}\left(x(1-t)^{-\frac{1}{k}};k\right)$$

[by use of relation (2.2.2)]

$$= (1-t)^{-(\alpha+1)/k} \exp\left(x\left[1-(1-t)^{-\frac{1}{k}}\right]\right) \sum_{j=0}^{\infty} c_j\, Y_{m+qj}^{\alpha}\left(x(1-t)^{-\frac{1}{k}};k\right) \cdot \Delta_j(y_1,\dots,y_N)\left(\frac{z\,t^q}{(1-t)^q}\right)^j$$

$$= (1-t)^{-(\alpha+1)/k} \exp\left(x\left[1-(1-t)^{-\frac{1}{k}}\right]\right) \times F\left[x(1-t)^{-\frac{1}{k}};y_1,\dots,y_N;\frac{zt^q}{(1-t)^q}\right]$$

which proves (2.6.1).

Proof of (2.6.2):

Consider

$$\sum_{n=0}^{\infty} Y_{m+n}^{\alpha-kn}(x;k)\,\Lambda_n(y_1,\dots,y_N;z)\,t^n$$

$$= \sum_{n=0}^{\infty} Y_{m+n}^{\alpha-kn}(x;k)\, t^n \sum_{j=0}^{[n/q]} \binom{m+n}{n-qj} c_j \Delta_j(y_1,\dots,y_N) z^j$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} Y_{m+n+qj}^{\alpha-kn-kqj}(x;k) \binom{m+n+qj}{n} t^{n+qj} \cdot c_j \Delta_j(y_1,\cdots,y_N) z^j$$

$$= \sum_{j=0}^{\infty} t^{qj} z^j c_j \Delta_j(y_1,\cdots,y_N) \sum_{n=0}^{\infty} \binom{m+n+qj}{n} Y_{m+n+qj}^{\alpha-kn-kqj}(x;k) t^n$$

$$= \sum_{j=0}^{\infty} t^{qj} z^j c_j \Delta_j(y_1,\cdots,y_N)\,(1+t)^{(\alpha-kqj-k+1)/k} \exp(x[1-(1+t)^{\frac{1}{k}}]) \times Y_{m+qj}^{\alpha-kqj}(x(1+t)^{\frac{1}{k}};k)$$

[by use of relation (2.3.2)]

$$= (1+t)^{(\alpha-k+1)/k} \exp(x[1-(1+t)^{\frac{1}{k}}]) \sum_{j=0}^{\infty} Y_{m+qj}^{\alpha-kqj}(x(1+t)^{\frac{1}{k}};k) \cdot c_j \left(\frac{t^q z}{(1+t)^q}\right)^j \Delta_j(y_1,\cdots,y_N)$$

$$= (1+t)^{(\alpha-k+1)/k} \exp(x[1-(1+t)^{\frac{1}{k}}]) \times G\left[x(1+t)^{\frac{1}{k}}, y_1,\cdots,y_N; \frac{zt^q}{(1+t)^q}\right]$$

which proves (2.6.2).

# CHAPTER-III

## PROPERTIES OF KONHAUSER POLYNOMIALS $Z_n^\alpha(x;k)$ :

### 3.1 INTRODUCTION:

The Konhauser polynomial $Z_n^\alpha(x;k)$ has been defined (Chapter I, ( 1.3.7 ) ) as

$$(3.1.1)\quad Z_n^\alpha(x;k) = \frac{\Gamma(1+\alpha+kn)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(1+\alpha+kj)},$$

where $\alpha > -1$, and k is a positive integer. In view of (3.1.1) $Z_n^\alpha(x;k)$ can alternatively be defined in terms of generalised hypergeometric functions ${}_pF_q$ (See Appendix) as

$$(3.1.2)\quad Z_n^\alpha(x;k) = \frac{(\alpha+1)_{kn}}{n!}\, {}_1F_k\left[-n; \frac{\alpha+1}{k}, \text{--------} \frac{\alpha+k}{k}; \left(\frac{x}{k}\right)^k\right].$$

In the present Chapter, we shall derive differential equations, recurrence relations, generating functions, multilinear generating functions and Laplace transforms associated with $Z_n^\alpha(x;k)$.

### 3.2 DIFFERENTIAL EQUATIONS:

$Z_n^\alpha(x;k)$ satisfies the following differential equation-

$$(3.2.1)\quad \left\{\prod_{j=1}^{k} (\delta+\alpha-k+j)\right\} \delta Z_n^\alpha(x;k) = x^k(\delta-kn)\, Z_n^\alpha(x;k); \quad \delta = x\frac{d}{dx},$$

which is equivalent to

$$(3.2.2)\quad D_x^k\left\{x^{\alpha+1} D_x Z_n^\alpha(x;k)\right\} = x^\alpha(xD_x-kn)\, Z_n^\alpha(x;k); \quad D=\frac{d}{dx}.$$

To prove (3.2.1), we shall use the well known differential equation for ${}_pF_q$ which is as follows [25] :

$$(3.2.3) \quad \left[\theta \prod_{j=1}^{q} (\theta+\beta_j-1) - z \prod_{j=1}^{p} (\theta+\alpha_j)\right]\omega = 0 ,$$

where

$$(3.2.4) \quad \omega = {}_pF_q\left[\alpha_1,\alpha_2,---;\alpha_p;\beta_1,----,\beta_q;z\right]$$

$$= \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_m z^m}{\prod_{j=1}^{q} (\beta_j)_m \, m!} ,$$

where $\beta_j \neq 0,-1,-2,----, \quad \forall j \in \{1,2,---,q\}$.

In equation (3.2.3), we set $p = 1$, $q = k$, $z = (x/k)^k$, $\theta = k^{-1}\delta$, $\delta = xD_x$, and comparing with equation (3.1.2), we get

$$\alpha_1 = -n, \; q = k, \; \beta_1 = \frac{\alpha+1}{k}, \; \beta_2 = \frac{\alpha+2}{k}, \; -----, \; \beta_k = \frac{\alpha+k}{k} ;$$

and (3.2.3) reduces to

$$k^{-1}\delta \prod_{j=1}^{k} \left(k^{-1}\delta + \frac{\alpha+j}{k} - 1\right)\omega = \left(\frac{x}{k}\right)^k (k^{-1}\delta - n)\omega$$

or $$\frac{\delta}{k} \prod_{j=1}^{k} \frac{(\delta+\alpha+j-k)}{k}\,\omega = \frac{x^k}{k^k}\left(\frac{\delta-nk}{k}\right)\omega$$

or $$\prod_{j=1}^{k} (\delta+\alpha+j-k)\frac{\delta\,\omega}{k\cdot k^k} = \frac{x^k (\delta-nk)\omega}{k^k\cdot k}$$

or $$\prod_{j=1}^{k} (\delta+\alpha+j-k)\delta\omega = x^k(\delta-nk)\omega$$

where $$\omega = {}_1F_k\left[-n;\frac{\alpha+1}{k},\frac{\alpha+2}{k},-----,\frac{\alpha+k}{k};\left(\frac{x}{k}\right)^k\right]$$

$$= \frac{n!}{(1+\alpha)_{kn}} Z_n^{\alpha}(x;k) .$$

As such, we get

$$\left\{\prod_{j=1}^{k} (\delta+\alpha-k+j)\right\}\delta Z_n^{\alpha}(x;k) = x^k(\delta-nk)Z_n^{\alpha}(x;k),$$

which proves (3.2.1)

From (1.5.5), for k = 0, we get

$$f(\delta+\alpha)\{g(x)\} = x^{-\alpha} f(\delta)\{x^{\alpha} g(x)\}, \quad \delta = x D_x .$$

Hence

$$(3.2.5) \quad \prod_{j=1}^{k} (\delta+\alpha-k+j)\{g(x)\} = x^{-\alpha} \prod_{j=1}^{k} (\delta-k+j)\{x^{\alpha} g(x)\}$$

$$= x^{k-\alpha} D^{k} \{x^{\alpha} g(x)\}$$

as $\delta(\delta-1) \cdots\cdots (\delta-k+1) = x^{k} D^{k}$.

Taking $g(x) = \delta Z_n^{\alpha}(x;k)$, we get

$$\left\{\prod_{j=1}^{k} (\delta+\alpha-k+j)\right\} \delta Z_n^{\alpha}(x;k) = x^{k-\alpha} D_x^{k} \{x^{\alpha} \delta Z_n^{\alpha}(x;k)\}$$

or $$D_x^{k} \{x^{\alpha+1} D_x Z_n^{\alpha}(x;k)\} = x^{\alpha} (x D_x - kn) Z_n^{\alpha}(x;k),$$

which proves (3.2.2).

## 3.3 RECURRENCE RELATIONS:

$Z_n^{\alpha}(x;k)$ satisfies the following recurrence formulas:

$$(3.3.1) \quad D_x Z_n^{\alpha}(x;k) = -k x^{k-1} Z_{n-1}^{\alpha+k}(x;k),$$

$$(3.3.2) \quad (x^{1-k} D_x)^m Z_n^{\alpha}(x;k) = (-k)^m Z_{n-m}^{\alpha+km}(x;k),$$

$$(3.3.3) \quad x D_x Z_n^{\alpha}(x;k) = (kn+\alpha) Z_n^{\alpha-1}(x;k) - \alpha Z_n^{\alpha}(x;k),$$

$$(3.3.4) \quad Z_n^{\alpha}(x;k) - Z_n^{\alpha-1}(x;k) = \frac{k\,\Gamma(kn+\alpha)}{\Gamma(k(n-1)+\alpha+1)} Z_{n-1}^{\alpha}(x;k),$$

$$(3.3.5) \quad x D_x Z_n^{\alpha}(x;k) = kn\, Z_n^{\alpha}(x;k) - \frac{k\,\Gamma(kn+\alpha+1)}{\Gamma(k(n-1)+\alpha+1} Z_{n-1}^{\alpha}(x;k)$$

(3.3.6) $x^k Z_n^{\alpha+k}(x;k) = (kn+\alpha+1)_k Z_n^{\alpha}(x;k) - (n+1) Z_{n+1}^{\alpha}(x;k)$ ,

(3.3.7) $k x^k Z_n^{\alpha+k}(x;k) = \alpha Z_{n+1}^{\alpha}(x;k) - (kn+\alpha+k) Z_{n+1}^{\alpha-1}(x;k)$ .

For this we use the well known result (See Appendix).

(3.3.8) $$D_z\left\{{}_pF_q[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z]\right\}$$

$$= \frac{\alpha_1,\ldots,\alpha_p}{\beta_1,\ldots,\beta_q}\,{}_pF_q[\alpha_1+1,\ldots,\alpha_p+1;\beta_1+1,\ldots,\beta_q+1;z].$$

Proof of (3.3.1) and (3.3.2):

Substituting p = 1, q = k, $z = \left(\frac{x}{k}\right)^k$, $D_z = \left(\frac{k}{x}\right)^{k-1} D_x$ in equation (3.3.8) and using (3.1.2), we get.

$$\left(\frac{k}{x}\right)^{k-1} D_x\,{}_1F_k\left[-n;\frac{\alpha+1}{k},\ldots,\frac{\alpha+k}{k};\left(\frac{x}{k}\right)^k\right]$$

$$= \frac{-n}{\prod_{j=1}^{k}\left(\frac{\alpha+j}{k}\right)}\,{}_1F_k\left[-n+1,\frac{\alpha+k+1}{k},\ldots,\frac{\alpha+k+k}{k};\left(\frac{x}{k}\right)^k\right]$$

or $$\frac{\left(\frac{k}{x}\right)^{k-1} D_x\, n!\, Z_n^{\alpha}(x;k)}{(\alpha+1)_{kn}} = \frac{-n(n-1)!\, Z_{n-1}^{\alpha+k}(x;k)}{\prod_{j=1}^{k}\left(\frac{\alpha+j}{k}\right)(\alpha+k+1)_{k(n-1)}}$$

or $$\frac{k^k D_x Z_n^{\alpha}(x;k)}{(\alpha+1)_{kn}} = \frac{-k x^{k-1}\cdot k^k Z_{n-1}^{\alpha+k}(x;k)}{(\alpha+1)_{kn}}$$

or $$D_x Z_n^{\alpha}(x;k) = -k x^{k-1} Z_{n-1}^{\alpha+k}(x;k),$$

which proves (3.3.1).

Rewriting (3.3.1) as

$$(x^{1-k}D)\, Z_n^{\alpha}(x;k) = (-k)\, Z_{n-1}^{\alpha+k}(x;k),$$

we get after repeating operation of $(x^{1-k}D)$, m times

$$(x^{1-k}D_x)^m Z_n^{\alpha}(x;k) = (-k)^m Z_{n-m}^{\alpha+km}(x;k),$$

which proves (3.3.2).

Proof of (3.3.3):

From (3.1.2)

$$Z_n^{\alpha}(x;k) = \frac{(\alpha+1)_{kn}}{n!}\,{}_1F_k\left[-n;\frac{\alpha+1}{k},\cdots,\frac{\alpha+k}{k};\left(\frac{x}{k}\right)^k\right]$$

$$= \frac{\Gamma(\alpha+1)(\alpha+1)_{kn}}{n!}\sum_{j=0}^{n}\frac{(-n)_j\,k^{kj}\,x^{kj}}{j!\,\Gamma(kj+\alpha+1)\,k^{kj}}\,.$$

Hence $x D_x Z_n^{\alpha}(x;k)$

$$= \frac{\Gamma(\alpha+1)(\alpha+1)_{kn}}{n!}\sum_{j=0}^{n}\frac{(-n)_j(kj)\,x^{kj}}{j!\,\Gamma(kj+\alpha+1)}\,.$$

Since $kj = (\alpha+kj)-\alpha$, we have

$$x D_x Z_n^{\alpha}(x;k) = \frac{\Gamma(\alpha+1)(\alpha+1)_{kn}}{n!}\sum_{j=0}^{n}\frac{(-n)_j(kj+\alpha)\,x^{kj}}{j!\,\Gamma(kj+\alpha+1)}$$

$$- \frac{\Gamma(\alpha+1)(\alpha+1)_{kn}}{n!}\sum_{j=0}^{n}\frac{(-n)_j\,\alpha\,x^{kj}}{j!\,\Gamma(kj+\alpha+1)}$$

$$= \frac{\Gamma(\alpha+1)(\alpha+1)_{kn}}{n!}\sum_{j=0}^{n}\frac{(-n)_j\,x^{kj}(\alpha+kj)}{j!\,\Gamma(kj+\alpha)(kj+\alpha)} - \alpha Z_n^{\alpha}(x;k)$$

$$= \frac{\Gamma(\alpha+kn+1)}{n!}\sum_{j=0}^{n}\frac{(-n)_j\,x^{kj}}{j!\,\Gamma(kj+\alpha)} - \alpha Z_n^{\alpha}(x;k)$$

$$= \frac{(\alpha+kn)\Gamma(\alpha+kn)}{n!}\sum_{j=0}^{n}\frac{(-n)_j\,x^{kj}}{j!\,\Gamma(kj+\alpha)} - \alpha Z_n^{\alpha}(x;k)$$

$$= \frac{(\alpha+kn)\,\Gamma\alpha\,(\alpha)_{kn}}{n!}\sum_{j=0}^{n}\frac{(-n)_j\,x^{kj}}{j!\,\Gamma(kj+\alpha)} - \alpha Z_n^{\alpha}(x;k)$$

$$= (\alpha+kn)\,Z_n^{\alpha-1}(x;k) - \alpha Z_n^{\alpha}(x;k)$$

i.e.

$$x D_x Z_n^{\alpha}(x;k) = (kn+\alpha)\,Z_n^{\alpha-1}(x;k) - \alpha Z_n^{\alpha}(x;k)\,,$$

which proves (3.3.3)

Proof of (3.3.4):

Consider

$$Z_n^{\alpha}(x;k) - Z_n^{\alpha-1}(x;k) = \sum_{j=0}^{n}\frac{(-n)_j\,x^{kj}}{j!\,n!}\left[\frac{\Gamma(\alpha+1)(\alpha+1)_{kn}}{\Gamma(kj+\alpha+1)} - \frac{\Gamma\alpha\,(\alpha)_{kn}}{\Gamma(kj+\alpha)}\right]$$

$$= \sum_{j=0}^{n}\frac{(-n)_j\,x^{kj}}{j!\,n!}\,\frac{\Gamma\alpha}{\Gamma(kj+\alpha)}\left[\frac{\alpha(\alpha+1)_{kn}}{(kj+\alpha)} - (\alpha)_{kn}\right]$$

$$= \sum_{j=0}^{n}\frac{(-n)_j\,x^{kj}}{j!\,n!}\,\frac{\Gamma\alpha}{\Gamma(kj+\alpha)}\left[\frac{(\alpha)_{kn+1}}{(kj+\alpha)} - (\alpha)_{kn}\right]$$

$$= \sum_{j=0}^{n}\frac{(-n)_j\,x^{kj}}{j!\,n!}\,\frac{\Gamma\alpha}{\Gamma(kj+\alpha)}\,(\alpha)_{kn}\left[\frac{\alpha+kn}{(kj+\alpha)} - 1\right]$$

$$= \sum_{j=0}^{n}\frac{(-n)_j\,x^{kj}}{j!\,n!}\,\frac{\Gamma\alpha}{\Gamma(kj+\alpha)}\,(\alpha)_{kn}\,\frac{k(n-j)}{(kj+\alpha)}$$

$$= \sum_{j=0}^{n-1}\frac{n(-n+1)_j\,x^{kj}\,k\,\Gamma(kn+\alpha)}{j!\,n!\,\Gamma(kj+\alpha+1)}$$

$$= \sum_{j=0}^{n-1}\frac{(-n+1)_j\,x^{kj}\,k\,\Gamma(kn+\alpha)\,\Gamma(k(n-1)+\alpha+1)}{(n-1)!\,j!\,\Gamma(kj+\alpha+1)\,\Gamma(k(n-1)+\alpha+1)}$$

$$= \sum_{j=0}^{n-1} \frac{(-n+1)_j\, x^{kj}\, k\, \Gamma(kn+\alpha)\, \Gamma(\alpha+1)\, (\alpha+1)_{k(n-1)}}{(n-1)!\; j!\; \Gamma(kj+\alpha+1)\; \Gamma(k(n-1)+\alpha+1)}$$

$$= \frac{k\,\Gamma(kn+\alpha)}{\Gamma(k(n-1)+\alpha+1)}\; \Gamma(\alpha+1)\,(\alpha+1)_{k(n-1)} \sum_{j=0}^{n-1} \frac{(-n+1)_j\, x^{kj}}{(n-1)!\; j!\; \Gamma(kj+\alpha+1)}$$

$$= \frac{k\,\Gamma(kn+\alpha)}{\Gamma(k(n-1)+\alpha+1)}\; Z_{n-1}^{\alpha}(x;k) \; ,$$

which proves (3.3.4)

<u>Proof of (3.3.5)</u>:

Substituting the value of $Z_n^{\alpha-1}(x;k)$ from (3.3.4) into equation (3.3.3) we get,

$$x D_x Z_n^{\alpha}(x;k) = (kn+\alpha)\, Z_n^{\alpha}(x;k) - \frac{k\,(kn+\alpha)\,\Gamma(kn+\alpha)}{\Gamma(k(n-1)+\alpha+1)}\, Z_{n-1}^{\alpha}(x;k) - \alpha\, Z_n^{\alpha}(x;k)$$

$$= kn\, Z_n^{\alpha}(x;k) - \frac{k\,\Gamma(kn+\alpha+1)}{\Gamma(k(n-1)+\alpha+1)}\, Z_{n-1}^{\alpha}(x;k)$$

which proves (3.3.5).

<u>Proof of (3.3.6)</u>:

Eliminating $x D_x Z_n^{\alpha}(x;k)$ between (3.3.5) and (3.3.1), we get

$$D_x Z_n^{\alpha}(x;k) = -k x^{k-1} Z_{n-1}^{\alpha+k}(x;k)$$

or

$$x D_x Z_n^{\alpha}(x;k) = -k x^{k} Z_{n-1}^{\alpha+k}(x;k)$$

$$-k x^{k} Z_{n-1}^{\alpha+k}(x;k) = kn\, Z_n^{\alpha}(x;k) - \frac{k\,\Gamma(kn+\alpha+1)}{\Gamma(k(n-1)+\alpha+1)}\, Z_{n-1}^{\alpha}(x;k)$$

or $x^{k} Z_{n-1}^{\alpha+k}(x;k) = \frac{\Gamma(kn+\alpha+1)}{\Gamma(k(n-1)+\alpha+1)} Z_{n-1}^{\alpha}(x;k) - n\, Z_{n}^{\alpha}(x;k)$

or $x^{k} Z_{n}^{\alpha+k}(x;k) = \frac{\Gamma(kn+k+\alpha+1)}{\Gamma(kn+\alpha+1)} Z_{n}^{\alpha}(x;k) - (n+1) Z_{n+1}^{\alpha}(x;k)$

or $x^{k} Z_{n}^{\alpha+k}(x;k) = (kn+\alpha+1)_{k} Z_{n}^{\alpha}(x;k) - (n+1) Z_{n+1}^{\alpha}(x;k)$,

which proves (3.3.6).

<u>Proof of (3.3.7)</u>:

Eliminating $x D_x Z_n^{\alpha}(x;k)$ between (3.3.3) and (3.3.1) we get

$$x D_x Z_n^{\alpha}(x;k) = -k x^{k} Z_{n-1}^{\alpha+k}(x;k)$$

or

$$-k x^{k} Z_{n-1}^{\alpha+k}(x;k) = (kn+\alpha) Z_{n}^{\alpha-1}(x;k) - \alpha Z_{n}^{\alpha}(x;k)$$

or

$$k x^{k} Z_{n-1}^{\alpha+k}(x;k) = \alpha Z_{n}^{\alpha}(x;k) - (kn+\alpha) Z_{n}^{\alpha-1}(x;k)$$

or

$$k x^{k} Z_{n}^{\alpha+k}(x;k) = \alpha Z_{n+1}^{\alpha}(x;k) - (kn+\alpha+k) Z_{n+1}^{\alpha-1}(x;k),$$

which proves (3.3.7).

## 3.4 GENERATING FUNCTIONS:

Following are the generating relations for $Z_n^{\alpha}(x;k)$:

(3.4.1) $$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\alpha+1)_{kn}} Z_n^{\alpha}(x;k) t^n = (1-t)^{-\lambda}\, {}_1F_k\left[\lambda; \frac{\alpha+1}{k}, \dots, \frac{\alpha+k}{k}; \frac{x^k t}{k^k(t-1)}\right],$$

(3.4.2) $$\sum_{n=0}^{\infty} Z_n^{\alpha}(x;k) \frac{t^n}{(\alpha+1)_{kn}} = e^{t}\, {}_0F_k\left[-; \frac{\alpha+1}{k}, \dots, \frac{\alpha+k}{k}; -\left(\frac{x}{k}\right)^{k} t\right],$$

(3.4.3) $$\sum_{n=0}^{\infty} \binom{m+n}{n} \frac{Z_{m+n}^{\alpha}(x;k)\, t^n}{(\alpha+1)_{k(m+n)}}$$

$$= \sum_{n=m}^{\infty} \binom{n}{m} \frac{t^{n-m}(-x^k)^n}{n!\,(\alpha+1)_{kn}}\, {}_1F_1\left[n+1; n-m+1; t\right].$$

To prove (3.4.1) and (3.4.2), we first prove the following relations:

(3.4.4) $$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}_{p+1}F_q\left[-n, \alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z\right] t^n = (1-t)^{-\lambda}\, {}_{p+1}F_q\left[\lambda, \alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; \frac{zt}{t-1}\right]_{t<1},$$

(3.4.5) $$\sum_{n=0}^{\infty} {}_{p+1}F_q\left[-n, \alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z\right] \frac{t^n}{n!} = e^t\, {}_pF_q\left[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; -zt\right].$$

Consider

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}_{p+1}F_q\left[-n, \alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z\right] t^n$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(\lambda)_n (-n)_j (\alpha_1, \ldots, \alpha_p)_j}{n!\,(\beta_1, \ldots, \beta_q)_j} \frac{z^j}{j!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(\lambda)_n (\alpha_1, \ldots, \alpha_p)_j}{(\beta_1, \ldots, \beta_q)_j} \frac{(-1)^j z^j}{(1)_{n-j}\, j!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\lambda)_{n+j} (\alpha_1, \ldots, \alpha_p)_j}{(\beta_1, \ldots, \beta_q)_j} \frac{(-1)^j z^j}{n!\, j!} t^{n+j}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\lambda+j)_n}{n!} \frac{t^n (\lambda)_j (-zt)^j}{j!\,(\beta_1, \ldots, \beta_q)_j} (\alpha_1, \ldots, \alpha_p)_j$$

$$= \sum_{j=0}^{\infty} \frac{(\lambda)_j (\alpha_1, \ldots, \alpha_p)_j (-zt)^j}{j!\,(\beta_1, \ldots, \beta_q)_j (1-t)^{\lambda+j}}$$

$$= (1-t)^{-\lambda} \sum_{j=0}^{\infty} \frac{(\lambda)_j (\alpha_1, \ldots, \alpha_p)_j}{j!\,(\beta_1, \ldots, \beta_q)_j} \frac{(-zt)^j}{(1-t)^j}$$

$$= (1-t)^{-\lambda} {}_{p+1}F_q\left[\lambda;\alpha_1,\text{-----},\alpha_p;\beta_1,\text{-----},\beta_q;-\frac{zt}{1-t}\right]$$

$$= (1-t)^{-\lambda} {}_{p+1}F_q\left[\lambda;\alpha_1,\text{-----},\alpha_p;\beta_1,\text{-----},\beta_q;\frac{zt}{t-1}\right].$$

This proves (3.4.4)

Hence for p = 0 and q = k and comparing with equation (3.1.2)

we get,

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\,{}_1F_k\left[-n;\frac{\alpha+1}{k},\text{------},\frac{\alpha+k}{k};\left(\frac{x}{k}\right)^k\right]t^n$$

$$= (1-t)^{-\lambda}\,{}_1F_k\left[\lambda;\frac{\alpha+1}{k},\text{---},\frac{\alpha+k}{k};\frac{x^k t}{k^k(t-1)}\right]$$

or $$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\alpha+1)_{kn}} Z_n^{\alpha}(x;k)\,t^n$$

$$= (1-t)^{-\lambda}\,{}_1F_k\left[\lambda;\frac{\alpha+1}{k},\text{-----},\frac{\alpha+k}{k};\frac{x^k t}{k^k(t-1)}\right],$$

which proves (3.4.1).

Now replacing t on both sides of (3.4.4) by $t/\lambda$ and taking

the limit as $\lambda\to\infty$, we get easily

$$\sum_{n=0}^{\infty} {}_{p+1}F_q\left[-n,\alpha_1,\text{---},\alpha_p;\beta_1,\text{-----},\beta_q;z\right]\frac{t^n}{n!}$$

$$= e^t\,{}_pF_q\left[\alpha_1,\text{---},\alpha_p;\beta_1,\text{----},\beta_q;-zt\right].$$

This proves (3.4.5).

Hence for p = 0 and q = k, we get, on comparision with equation

(3.1.2),

$$\sum_{n=0}^{\infty} {}_1F_k\left[-n;\frac{\alpha+1}{k},\text{-----},\frac{\alpha+k}{k};\left(\frac{x}{k}\right)^k\right]\frac{t^n}{n!}$$

$$= e^t\,{}_0F_k\left[-;\frac{\alpha+1}{k},\text{---},\frac{\alpha+k}{k};-\left(\frac{x}{k}\right)^k t\right]$$

or $$\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;k)\, t^n}{(\alpha+1)_{kn}} = e^t \, {}_0F_k\left[-;\frac{\alpha+1}{k},\cdots,\frac{\alpha+k}{k};-\left(\frac{x}{k}\right)^k t\right],$$

which proves (3.4.2).

<u>Proof of (3.4.3)</u>:

Consider the double series

$$\sum_{m=0}^{\infty} z^m \sum_{n=0}^{\infty} \binom{m+n}{n} Z_{m+n}^{\alpha}(x;k) \frac{t^n}{(\alpha+1)_{k(m+n)}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{Z_n^{\alpha}(x;k)}{(\alpha+1)_{kn}} \binom{n}{m} t^{n-m} z^m = \sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;k)}{(\alpha+1)_{kn}} \sum_{m=0}^{n} \binom{n}{m} t^{n-m}\cdot z^m$$

$$= \sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;k)\,(z+t)^n}{(\alpha+1)_{kn}}$$

$$= e^{z+t}\, {}_0F_k\left[-;\frac{\alpha+1}{k},\cdots,\frac{\alpha+k}{k};-\left(\frac{x}{k}\right)^k (z+t)\right]$$

[by (3.4.2)]

$$= \sum_{\nu=0}^{\infty} \frac{(z+t)^{\nu}}{\nu!} \sum_{n=0}^{\infty} \frac{(-x^k)^n (z+t)^n}{n!\,(\alpha+1)_{kn}}$$

$$= \sum_{n,\nu=0}^{\infty} \frac{(-x^k)^n}{\nu!\,n!\,(\alpha+1)_{kn}} (z+t)^{n+\nu}$$

$$= \sum_{n,\nu=0}^{\infty} \frac{(-x^k)^n}{\nu!\,n!\,(\alpha+1)_{kn}} \sum_{m=0}^{n+\nu} \binom{n+\nu}{m} z^m t^{n+\nu-m}$$

$$= \sum_{m=0}^{\infty} z^m \sum_{n+\nu \geq m}^{\infty} \binom{n+\nu}{m} \frac{t^{n-m}}{n!} \frac{(-x^k)^n}{(\alpha+1)_{kn}} \frac{t^{\nu}}{\nu!}$$

$$= \sum_{m=0}^{\infty} z^m \sum_{n=m}^{\infty} \sum_{\nu=0}^{\infty} \frac{(n+\nu)!\, t^{n-m} (-x^k)^n t^\nu}{m!\,(n+\nu-m)!\, n!\, \nu!\, (\alpha+1)_{kn}}$$

$$= \sum_{m=0}^{\infty} z^m \sum_{n=m}^{\infty} \sum_{\nu=0}^{\infty} \frac{(n+\nu)!\,(n-m)!\, n!\, t^{n-m} (-x^k)^n t^\nu}{m!\,(n+\nu-m)!\,(n-m)!\,(n!)^2\, \nu!\, (\alpha+1)_{kn}}$$

$$= \sum_{m=0}^{\infty} z^m \sum_{n=m}^{\infty} \binom{n}{m} \frac{t^{n-m} (-x^k)^n}{n!\,(\alpha+1)_{kn}} \sum_{\nu=0}^{\infty} \frac{(n+1)_\nu t^\nu}{\nu!\,(n-m+1)_\nu} .$$

Since $\dfrac{(n-m)!}{(n+\nu-m)!} = \dfrac{1}{(n-m+1)_\nu}$ , $\dfrac{(n+\nu)!}{n!} = (n+1)_\nu$

$$= \sum_{m=0}^{\infty} z^m \sum_{n=m}^{\infty} \binom{n}{m} \frac{t^{n-m} (-x^k)^n}{n!\,(\alpha+1)_{kn}} \, {}_1F_1[n+1; n-m+1; t]$$

and on equating the coefficients of $z^m$ both sides we get,

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Z_{m+n}^{\alpha}(x;k) \frac{t^n}{(\alpha+1)_{k(m+n)}}$$

$$= \sum_{n=m}^{\infty} \binom{n}{m} \frac{t^{n-m} (-x^k)^n}{n!\,(\alpha+1)_{kn}} \, {}_1F_1[n+1; n-m+1; t] ,$$

which proves (3.4.3).

Clearly, for m = 0, (3.4.3) reduces to (3.4.1). As a special case, when k = 1, (3.4.3) reduces to the generating function for Laguerre polynomials, to be explicit, from (3.4.3), after simple series manupulation, we get

$$(3.4.6)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha)}(x) \frac{t^n}{(1+\alpha)_n} = \binom{\alpha+m}{m} e^x \Psi_2[\alpha+m+1; \alpha+1, \alpha+1; t, -x],$$

where $\Psi_2$ is (Humbert's) confluent hypergeometric function of two variables, defined by [2].

$$(3.4.7)\quad \Psi_2[a; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(c)_m (c')_n} \frac{x^m}{m!} \cdot \frac{y^n}{n!}.$$

Interestingly (3.4.6) is a special case of analogous generating relation due to H.M. Srivastava [32],

$$(3.4.8)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha)}(x) \frac{t^n}{(\mu)_n}$$

$$= \binom{\alpha+m}{m} e^x \Psi_2[\alpha+m+1; \mu, \alpha+1; t, -x].$$

Clearly (3.4.8) reduces to (3.4.6) for $\mu = 1+\alpha$.

If we replace t by $\mu t$ in (3.4.8) and take the limit as $\mu \to \infty$, we get the familiar generating relation for $L_n^{\alpha}(x)$ as

$$(3.4.9)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha)}(x) t^n$$

$$= (1-t)^{-m-\alpha-1} \exp\left(\frac{-xt}{1-t}\right) \cdot L_n^{(\alpha)}\left(\frac{x}{1-t}\right), \quad m = 0, 1, 2, \cdots$$

This result also follows from (2.2.2) for $k = 1$.

It is interesting to note that the generating relation (3.4.1) is due to Genin et. Calvez [17], while (3.4.2) was given earlier by Srivastava [33].

## 3.5 FINITE SUMMATION FORMULAS:

Following are the finite summation formulas for $Z_n^{\alpha}(x;k)$

(3.5.1) $$Z_n^{\alpha}(x;k) = \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^{n} \binom{\alpha+kn}{kj} \frac{(kj)!}{j!} \left(\frac{y^k - x^k}{x^k}\right)^j Z_{n-j}^{\alpha}(y;k),$$

(3.5.2) $$Z_n^{\alpha}(x;k) = \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^{n} \binom{\alpha+kn}{kn-kj} \frac{(kn-kj)!}{(n-j)!} \left(\frac{y^k - x^k}{x^k}\right)^{n-j} Z_j^{\alpha}(y;k),$$

(3.5.3) $$Z_n^{\alpha}(\mu x;k) = \sum_{j=0}^{n} \binom{kn+\alpha}{kj} \frac{(kj)!}{j!} \mu^{k(n-j)} (1-\mu^k)^j Z_{n-j}^{\alpha}(x;k).$$

Proof of (3.5.1):

Multiplying both sides by $\frac{t^n}{(1+\alpha)_{kn}}$ and summing for n, we have

$$\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;k)\, t^n}{(1+\alpha)_{kn}} = \sum_{n=0}^{\infty} \frac{t^n}{(1+\alpha)_{kn}} \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^{n} \binom{\alpha+kn}{kj} \frac{(kj)!}{j!} \left(\frac{y^k - x^k}{x^k}\right)^j Z_{n-j}^{\alpha}(y;k).$$

Now RHS

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^{n+j}}{(1+\alpha)_{k(n+j)}} \left(\frac{x}{y}\right)^{k(n+j)} \binom{\alpha+k(n+j)}{kj} \frac{(kj)!}{j!} \left(\frac{y^k - x^k}{x^k}\right)^j Z_n^{\alpha}(y;k)$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{t^{n+j}(1+\alpha+kn)_{jk}}{(1+\alpha)_{kn}(1+\alpha+kn)_{kj}} \left(\frac{x}{y}\right)^{k(n+j)} \left(\frac{y^k-x^k}{x^k}\right)^j Z_n^\alpha(y;k)$$

$$= \sum_{n=0}^{\infty} \frac{\left[t\left(\frac{x}{y}\right)^k\right]^n Z_n^\alpha(y;k)}{(1+\alpha)_{kn}} \sum_{j=0}^{\infty} \frac{\left[t\left(\frac{x}{y}\right)^k\right]^j \left(\frac{y^k-x^k}{x^k}\right)^j}{j!}$$

$$= e^{t\left(\frac{x}{y}\right)^k} {}_0F_k\left[-;\frac{\alpha+1}{k},\ldots,\frac{\alpha+k}{k};-\left(\frac{y}{k}\right)^k t\left(\frac{x}{y}\right)^k\right]$$

$$\times \exp\left[t\cdot\frac{x^k}{y^k}\cdot\frac{y^k-x^k}{x^k}\right]$$

$$= \exp\left[t\frac{x^k}{y^k}+t-t\frac{x^k}{y^k}\right]\cdot {}_0F_k\left[-;\frac{\alpha+1}{k},\ldots,\frac{\alpha+k}{k};-\left(\frac{x}{k}\right)^k t\right]$$

$$= e^t {}_0F_k\left[-;\frac{\alpha+1}{k},\ldots,\frac{\alpha+k}{k};-\left(\frac{x}{k}\right)^k t\right]$$

$$= \text{LHS}.$$

Proof of (3.5.2):

Since $\sum_{j=0}^{n} a_j = \sum_{j=0}^{n} a_{n-j}$, the equation (3.5.1) transforms to

$$Z_n^\alpha(x;k) = \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^{n} \left(\frac{y^k-x^k}{x^k}\right)^{n-j} \binom{\alpha+kn}{kn-kj} \frac{(kn-kj)!}{(n-j)!} Z_j^\alpha(y;k),$$

which proves (3.5.2).

Proof of (3.5.3):

Substituting $x = \mu x$, and $y = x$ in equation (3.5.1), we have

$$Z_n^\alpha(\mu x;k) = \left(\frac{\mu x}{x}\right)^{kn} \sum_{j=0}^{n} \binom{kn+\alpha}{kj} \frac{(kj)!}{j!} \left(\frac{x^k-(\mu x)^k}{(\mu x)^k}\right)^j Z_{n-j}^\alpha(x;k)$$

$$= \sum_{j=0}^{n} \binom{kn+\alpha}{kj} \frac{(kj)!}{j!} \mu^{k(n-j)}(1-\mu^k)^j Z_{n-j}^\alpha(x;k),$$

which proves (3.5.3). This result can be used as a multiplication formula for $Z_n^{\alpha}(x;k)$.

## 3.6 LAPLACE TRANSFORMS:

Following are the Laplace transforms for $Z_n^{\alpha}(x;k)$:

(3.6.1) $\mathcal{L}\{t^{\beta} Z_n^{\alpha}(xt;k):s\}$

$$= \frac{(\alpha+1)_{kn}\Gamma(\beta+1)}{s^{\beta+1}\, n!} \times {}_{k+1}F_{k}\left[-n, \frac{\beta+1}{k}, \ldots, \frac{\beta+k}{k}; \frac{\alpha+1}{k}, \ldots, \frac{\alpha+k}{k}; \left(\frac{x}{s}\right)^{k}\right],$$

provided that $R_e(s) > 0$ and $R_e(\beta) > -1$.

(3.6.2) $$\mathcal{L}\{t^{\alpha} Z_n^{\alpha}(xt;k):s\} = \frac{\Gamma(kn+\alpha+1)}{s^{kn+\alpha+1}\, n!}(s^k - x^k)^n.$$

The Laplace transform of a function f is defined as

(3.6.3) $$\mathcal{L}\{f(t):s\} = \int_0^{\infty} e^{-st} f(t)\, dt \qquad R_e(s-\sigma) > 0$$

Proof of (3.6.1):

$\mathcal{L}\{t^{\beta} Z_n^{\alpha}(xt;k):s\}$

$$= \sum_{j=0}^{n} \frac{(-n)_j\, x^{kj}}{\prod_{i=1}^{k}\left(\frac{\alpha+i}{k}\right)_j k^{kj}} \cdot \frac{(\alpha+1)_{kn}}{n!\, j!} \int_0^{\infty} e^{-st}\, t^{\beta+kj}\, dt.$$

Since $$\int_0^{\infty} e^{-st}\, t^{\beta+kj}\, dt = (\beta+kj)!/s^{\beta+kj+1},$$

Hence

$\alpha\{t^{\beta} Z_n^{\alpha}(xt;k):s\}$

$$= \frac{(\alpha+1)_{kn}}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{\prod_{i=1}^{k}\left(\frac{\alpha+i}{k}\right)_j} \frac{(\beta+kj)!}{s^{\beta+kj+1}} \frac{x^{kj}}{j!\,k^{kj}}$$

$$= \frac{(\alpha+1)_{kn}}{s^{\beta+1}\,n!} \sum_{j=0}^{n} \frac{(-n)_j\, x^{kj}\, \Gamma(\beta+kj+1)}{\prod_{i=1}^{k}\left(\frac{\alpha+i}{k}\right)_j s^{kj} k^{kj} j!}$$

$$= \frac{(\alpha+1)_{kn}}{s^{\beta+1}\,n!} \sum_{j=0}^{n} \frac{(-n)_j\, x^{kj}\, \Gamma(\beta+1)\, k^{kj} \left(\frac{\beta+1}{k}\right)_j ; \ldots\ldots \left(\frac{\beta+k}{k}\right)_j}{\prod_{i=1}^{k}\left(\frac{\alpha+i}{k}\right)_j k^{kj} s^{kj} j!}$$

$$= \frac{(\alpha+1)_{kn}\,\Gamma(\beta+1)}{s^{\beta+1}\,n!} \times {}_{k+1}F_k\left[-n; \frac{\beta+1}{k}, \ldots\ldots, \frac{\beta+k}{k}; \frac{\alpha+1}{k}, \ldots \ldots\ldots, \frac{\alpha+k}{k}; \left(\frac{x}{s}\right)^k\right],$$

which proves (3.6.1).

<u>Proof of (3.6.2):</u>

Substituting $\beta=\alpha$ in equation (3.6.1)

$$\alpha\{t^{\alpha} Z_n^{\alpha}(xt;k):s\} = \frac{(\alpha+1)_{kn}\Gamma(\alpha+1)}{s^{\alpha+1}\,n!} \times {}_{k+1}F_k\left[-n; \frac{\alpha+1}{k}, \ldots\ldots, \frac{\alpha+k}{k}; \frac{\alpha+1}{k}, \ldots\ldots, \frac{\alpha+k}{k}; \left(\frac{x}{s}\right)^k\right]$$

$$= \frac{\Gamma(kn+\alpha+1)}{s^{\alpha+1}\,n!}\, {}_1F_0\left[-n; -; \left(\frac{x}{s}\right)^k\right]$$

$$= \frac{\Gamma(kn+\alpha+1)}{s^{\alpha+1+kn}\,n!} (s^k - x^k)^n,$$

as

$${}_1F_0(a;-;z) = (1-z)^{-a}.$$

This proves (3.6.2).

# CHAPTER-IV

## MULTILINEAR GENERATING RELATIONS

### 4.1 INTRODUCTION:

In the present Chapter, a general multilinear generating function for the polynomials $G_n^{(\alpha)}(x, h, p, k)$ has been considered. From this we shall derive as special cases, generating relations for Konhauser polynomials. So we state the following theorem:

THEOREM- For a bounded multiple sequence $\{\Lambda(n_1, \ldots, n_r)\}$ of arbitrary complex numbers, let

$$(4.1.1)\quad H[n_1, \ldots, n_r; y_1, \ldots, y_r] = \sum_{j_1=0}^{[n_1/m_1]} \cdots \sum_{j_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 j_1}}{j_1!} \cdots \frac{(-n_r)_{m_r j_r}}{j_r!} \cdot \Lambda(j_1, \ldots, j_r)\, y_1^{j_1} \cdots y_r^{j_r},$$

where $m_1, \ldots, m_r$ are positive integers. Also let $\Delta_r$ be defined by

$$(4.1.2)\quad \Delta_r = 1 - \sum_{i=1}^{r} u_i \,, \quad r = 1, 2, 3, \ldots$$

Then, for every nonnegative integer m,

$$(4.1.3)\quad \sum_{n_1, \ldots, n_r=0}^{\infty} (m+n_1+\cdots+n_r)!\, G_{m+n_1+\cdots+n_r}^{(\alpha)}(x, h, p, k) \cdot H[n_1, \ldots, n_r; y_1, \ldots, y_r] \frac{(u_1/k)^{n_1}}{n_1!} \cdots \frac{(u_r/k)^{n_r}}{n_r!}$$

$$= k^m \exp(px^h)\, \Delta_r^{-m-\alpha/k} \times$$

$$\times \sum_{n,n_1,\cdots n_r=0}^{\infty} \left(\frac{hn+\alpha}{k}\right)_{m+m_1n_1+\cdots+m_rn_r} \left(\frac{1}{n!}\right) \Lambda(n_1,\cdots,n_r)$$

$$\left(\frac{-px^h}{\Delta_r^{h/k}}\right)^n \cdot \prod_{i=1}^{r} \left\{\frac{[(-u_i|\Delta_r)^{m_i} y_i]^{n_i}}{n_i!}\right\}, \quad k \neq 0,$$

provided that the multiple series on the right-hand side of (4.1.3) has a meaning, and

(4.1.4) $\quad |u_1 + \cdots\cdots + u_r| < 1$ .

4.2 <u>Proof of the theorem</u> :

For convenience, let $\Omega(u_1,\cdots\cdots,u_r)$ denote the left-hand side of (4.1.3), and set

(4.2.1) $\quad N = n_1 + \cdots\cdots + n_r$ and $J = m_1j_1 + \cdots\cdots + m_rj_r$.

Applying the explicit representation (2.1.1) and the definition (4.1.1), we find that

(4.2.2) $$\Omega(u_1,\cdots\cdots,u_r) = k^m \sum_{n_1,\cdots,n_r=0}^{\infty} (m+n_1+\cdots+n_r)!\, \frac{k^{n_1+n_2+\cdots+n_r}}{(m+n_1+n_2+\cdots+n_r)}$$

$$\cdot \sum_{j=0}^{m+n_1+\cdots+n_r} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+n_1+\cdots+n_r}$$

$$\cdot \sum_{j_1=0}^{[n_1/m_1]} \cdots\cdots \sum_{j_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1j_1}}{j_1!} \cdots\cdots \frac{(-n_r)_{m_rj_r}}{j_r!} \Lambda(j_1,\cdots,j_r)$$

$$\cdot\, y_1^{j_1}, \cdots\cdots, y_r^{j_r} \left(\frac{(u_1|k)^{n_1}}{n_1!} \cdots\cdots \frac{(u_r|k)^{n_r}}{n_r!}\right).$$

$$= k^m \sum_{n_1,\dots,n_r=0}^{\infty} u_1^{n_1}, \dots, u_r^{n_r} \sum_{j=0}^{m+N} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+N}$$

$$\cdot \prod_{i=1}^{r} \left\{ \sum_{j_i=0}^{[n_i/m_i]} \frac{[(-1)^{m_i} y_i]^{j_i}}{j_i!\,(n_i - m_i j_i)!} \right\} \Lambda(j_1,\dots,j_r) .$$

Since $(1)_{n-mj} = \dfrac{(-1)^{mj}(1)_n}{(-n)_{mj}}$,

$\Omega(u_1,\dots,u_r)$

$$= k^m \sum_{j_1,\dots,j_r=0}^{\infty} \prod_{i=1}^{r} \left\{ \frac{[(-1)^{m_i} y_i]^{j_i}}{j_i!\,n_i!} \right\} \Lambda(j_1,\dots,j_r)$$

$$\cdot \sum_{n_1,\dots,n_r=0}^{\infty} \left( \prod_{i=1}^{r} [u_i^{n_i+m_i j_i}] \right) \sum_{j=0}^{m+N+J} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+N+J}$$

$$= k^m \sum_{j_1,\dots,j_r=0}^{\infty} \Lambda(j_1,\dots,j_r) \prod_{i=1}^{r} \left\{ \frac{[(-u_i)^{m_i} y_i]^{j_i}}{j_i!} \right\} \sum_{n_1,\dots,n_r=0}^{\infty} \frac{u_1^{n_1}}{n_1!}, \dots, \frac{u_r^{n_r}}{n_r!}$$

$$\cdot \sum_{j=0}^{m+N+J} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+N+J} .$$

Now we appeal to the well known series identity.

$$(4.2.3) \quad \sum_{n_1,\dots,n_r=0}^{\infty} f(n_1+\dots+n_r) \frac{u_1^{n_1}}{n_1!} \cdots \frac{u_r^{n_r}}{n_r!} = \sum_{n=0}^{\infty} f(n) \frac{(u_1+\dots+u_r)^n}{n!} ,$$

the equation (4.2.2) becomes

$$(4.2.4) \quad \Omega(u_1,\dots,u_r) = k^m \sum_{n,j_1,\dots,j_r=0}^{\infty} \frac{(u_1+\dots+u_r)^n}{n!} \Lambda(j_1,\dots,j_r)$$

$$\prod_{i=1}^{r} \left\{ \frac{[(-u_i)^{m_i} y_i]^{j_i}}{j_i!} \right\} \sum_{j=0}^{m+n+J} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+n+J}$$

Where J is defined as before, by (4.2.1).

The innermost sum in (4.2.4) is the $j^{th}$ difference of a polynomial of degree m + n + j in $\alpha$ ; it is nill when j > m + n + J. Thus we have

$$\sum_{j=0}^{m+n+J} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+n+J}$$

$$= \sum_{l=0}^{\infty} \left(\frac{hl+\alpha}{k}\right)_{m+n+J} \left(\frac{-px^h)^l}{l!}\right) \sum_{j=0}^{\infty} \frac{(px^h)^j}{j!}$$

$$= \exp(px^h) \sum_{l=0}^{\infty} \left(\frac{hl+\alpha}{k}\right)_{m+n+J} \frac{(-px^h)^l}{l!}.$$

Substituting this expression in (4.2.4) and applying the binomial expansion to sum the resulting n - series, we get

$$(4.2.5)\quad \Omega(v_1,\dots;v_r) = k^m \sum_{n,j_1,\dots,j_r=0}^{\infty} \frac{(v_1+\dots+v_r)^n}{n!} \Lambda(j_1,\dots,j_r)$$

$$\prod_{i=1}^{r} \left\{ \frac{[(-v_i)^{m_i} y_i]^{j_i}}{j_i!} \right\} \exp(px^h) \sum_{l=0}^{\infty} \left(\frac{hl+\alpha}{k}\right)_{m+n+J} \frac{(-px^h)^l}{l!}.$$

Case I : when r = 1,

$$(4.2.6)\quad \Omega(v_1) = k^m \exp(px^h) \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \frac{v_1^n}{n!} \Lambda(j_1)$$

$$\cdot \left\{ \frac{[(-v_1)^{m_1} y_1]^{j_1}}{j_1!} \right\} \sum_{l=0}^{\infty} \left(\frac{hl+\alpha}{k}\right)_{m+n+m_1 j_1} \frac{(-px^h)^l}{l!}$$

$$= k^{m} \exp(px^{h}) \sum_{l,j_1=0}^{\infty} \Lambda(j_1) \left\{ \frac{[(-u_1)^{m_1} y_1]^{j_1}}{j_1!} \right\} \frac{(-px^{h})^{l}}{l!} \left(\frac{hl+\alpha}{k}\right)_{m+m_1 j_1}$$

$$\cdot \sum_{n=0}^{\infty} \frac{u_1^{n}}{n!} \left(\frac{hl+\alpha}{k} + m + m_1 j_1\right)_{n}$$

$$= k^{m} \exp(px^{h}) \sum_{l,j_1=0}^{\infty} \Lambda(j_1) \left\{ \frac{[(-u_1)^{m_1} y_1]^{j_1}}{j_1!} \right\} \frac{(-px^{h})^{l}}{l!} \left(\frac{hl+\alpha}{k}\right)_{m+m_1 j_1}$$

$$\cdot (1-u_1)^{-\left(\frac{hl+\alpha}{k}\right) - m - m_1 j_1}$$

$$= k^{m} \exp(px^{h}) \sum_{l,j_1=0}^{\infty} \Lambda(j_1) \left\{ \frac{[(-\frac{u_1}{\Delta_1})^{m_1} y_1]^{j_1}}{j_1!} \right\} \left(\frac{1}{l!}\right) \left(\frac{-px^{h}}{\Delta_1^{h/k}}\right)^{l}$$

$$\cdot \Delta_1^{-m-\frac{\alpha}{k}} \left(\frac{hl+\alpha}{k}\right)_{m+m_1 j_1} .$$

<u>Case II</u>: When $r = 2$,

$$\Omega(u_1, u_2) = k^{m} \exp(px^{h}) \sum_{n=0}^{\infty} \sum_{j_1,j_2=0}^{\infty} \frac{(u_1+u_2)^{n}}{n!} \Lambda(j_1, j_2) \prod_{i=1}^{2} \left\{ \frac{[(-u_i)^{m_i} y_i]^{j_i}}{j_i!} \right\}$$

$$\cdot \sum_{l=0}^{\infty} \left(\frac{hl+\alpha}{k}\right)_{m+n+m_1 j_1 + m_2 j_2} \frac{(-px^{h})^{l}}{l!}$$

$$= k^{m} \exp(px^{h}) \sum_{l=0}^{\infty} \sum_{j_1,j_2=0}^{\infty} \Lambda(j_1, j_2) \left(\frac{hl+\alpha}{k}\right)_{m+m_1 j_1 + m_2 j_2}$$

$$\cdot \frac{(-px^{h})^{l}}{l!} \prod_{i=1}^{2} \left\{ \frac{[(-u_i)^{m_i} y_i]^{j_i}}{j_i!} \right\} \sum_{n=0}^{\infty} \frac{(u_1+u_2)^{n}}{n!} \left(\frac{hl+\alpha}{k} + m + m_1 j_1 + m_2 j_2\right)_{n}$$

$$= k^{m} \exp(px^{h}) \sum_{l,j_1,j_2=0}^{\infty} \Lambda(j_1, j_2) \cdot$$

$$\cdot \prod_{i=1}^{2} \left\{ \frac{[(-u_i)^{m_i} y_i]^{j_i}}{j_i\,!} \right\} \left(\frac{1}{\ell!}\right) (-px^h)^{\ell} \left(\frac{h\ell+\alpha}{k}\right)_{m+m_1j_1+m_2j_2}$$

$$\cdot \left(1-(u_1+u_2)\right)^{-\left(\frac{h\ell+\alpha}{k}\right)-m-m_1j_1-m_2j_2}$$

$$= k^m \exp(px^h) \sum_{\ell,j_1,j_2=0}^{\infty} \Lambda(j_1,j_2) \prod_{i=1}^{2} \left\{ \frac{[(-u_i/\Delta_i)^{m_i} y_i]^{j_i}}{j_i\,!} \right\}$$

$$\cdot \left(\frac{1}{\ell!}\right) \left(\frac{-px^h}{\Delta_2^{h/k}}\right)^{\ell} \cdot \Delta_2^{-m-\alpha/k} \left(\frac{h\ell+\alpha}{k}\right)_{m+m_1j_1+m_2j_2}.$$

And so in general, we finally obtain

$$(4.2.7) \quad \Omega(u_1,\cdots,u_r) = k^m \exp(px^h)\, \Delta_r^{-m-\alpha/k} \sum_{\ell,j_1,\cdots,j_r=0}^{\infty} \left(\frac{h\ell+\alpha}{k}\right)_{m+J}$$

$$\cdot \left(\frac{1}{\ell!}\right) \Lambda(j_1,\cdots,j_r) \left(\frac{-px^h}{\Delta_r^{h/k}}\right)^{\ell} \prod_{i=1}^{r} \left\{ \frac{[(-u_i/\Delta_i)^{m_i} y_i]^{j_i}}{j_i\,!} \right\}, \quad k \neq 0,$$

where $\Delta_r$ and J are given by (4.1.2) and (4.2.1) respectively. And the inequality in (4.1.4) is assumed to hold the right-hand side of (4.1.3) and (4.2.7) are essentially the same. This evidently completes the proof of our theorem.

## 4.3 GENERALISATION OF THE THEOREM:

For a bounded multiple sequence $\{\Lambda(n_1,\cdots,n_r)\}$ of arbitrary complex numbers, Let

(4.3.1) $H[n_1,\cdots,n_r; y_1,\cdots,y_r]$

$$= \sum_{j_1=0}^{[n_1/m_1]} \cdots \sum_{j_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 j_1}}{j_1!} \cdots \frac{(-n_r)_{m_r j_r}}{j_r!},$$

where $m_1,\cdots,m_r$ are positive integers. Also let $\Delta_r$ be defined by (4.1.2)

(4.3.2) $$\sum_{n_1,\cdots,n_r=0}^{\infty} (m+n_1+\cdots+n_r)!\, F^{(\alpha)}_{m+n_1+\cdots+n_r}(x,h,p,k) \cdot H[n_1,\cdots,n_r; y_1,\cdots,y_r] \frac{(u_1/k)^{n_1}}{n_1!} \cdots \frac{(u_r/k)^{n_r}}{n_r!}$$

$$= k^m \exp(px^h)\, \Delta_r^{-m-\alpha/k} \sum_{n,n_1,\cdots,n_r=0}^{\infty} \left(\frac{hn+\alpha}{k}\right)_{m+m_1n_1+\cdots+m_rn_r}$$

$$\cdot \frac{(\xi)_n}{n!}\, \Delta(n_1,\cdots,n_r) \left(\frac{-px^h}{\Delta_r^{h/k}}\right)^n \prod_{i=1}^{r} \left\{ \frac{[(-u_i/\Delta_r)^{m_i} y_i]^{n_i}}{n_i!} \right\},$$

where in terms of the bounded sequence $(\xi_n)$ of arbitrary complex numbers

(4.3.3) $$F_n^{(\alpha)}(x,h,p,k) = \frac{k^n}{n!} \sum_{j=0}^{\infty} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \xi_l \left(\frac{hl+\alpha}{k}\right)_n .$$

Proof: For convenience, let $\Omega(u_1,\cdots,u_r)$ denote the left hand side of (4.3.2), and applying the explicit representation (4.3.3) and the definition (4.3.1) we find that

(4.3.4) $$\Omega(u_1,\cdots,u_r) = \sum_{n_1,\cdots,n_r=0}^{\infty} (m+n_1+\cdots+n_r)!\, \frac{k^{m+n_1+\cdots+n_r}}{(m+n_1+\cdots+n_r)!}$$

$$\cdot \sum_{j=0}^{m+n_1+\cdots+n_r} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+n_1+\cdots+n_r} (\xi)_l$$

$$\cdot \sum_{j_1=0}^{[n_1/m_1]} \cdots\cdots \sum_{j_r=0}^{[n_r/m_r]} (-n_1)_{m_1 j_1} \cdots\cdots (-n_r)_{m_r j_r} \Lambda(j_1, \ldots, j_r)$$

$$\cdot y_1^{j_1} \cdots\cdots y_r^{j_r} \left( \frac{(u_1 k)^{n_1}}{n_1!} \cdots\cdots \frac{(u_r k)^{n_r}}{n_r!} \right)$$

$$= k^m \sum_{n_1,\ldots n_r=0}^{\infty} u_1^{n_1} \cdots\cdots u_r^{n_r} \sum_{j=0}^{m+N} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} (\xi)_l \cdot$$

$$\cdot \left(\frac{hl+\alpha}{k}\right)_{m+N} \prod_{i=1}^{r} \left\{ \sum_{j_i=0}^{[n_i/m_i]} \frac{[(-1)^{m_i} y_i]^{j_i}}{j_i!(n_i - m_i j_i)!} \right\} \Lambda(j_1, \ldots, j_r)$$

$$= k^m \sum_{j_1,\ldots,j_r=0}^{\infty} \prod_{i=1}^{r} \left\{ \frac{[(-1)^{m_i} y_i]^{j_i}}{j_i!\ n_i!} \right\} \Lambda(j_1, \ldots, j_r) \cdot$$

$$\cdot \sum_{n_1,\ldots,n_r=0}^{\infty} \left( \prod_{i=1}^{r} [u_1^{n_i + m_i j_i}] \right) \sum_{j=0}^{m+N+J} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} (\xi)_l$$

$$\cdot \left(\frac{hl+\alpha}{k}\right)_{m+N+J}$$

$$= k^m \sum_{j_1,\ldots,j_r=0}^{\infty} \Lambda(j_1, \ldots, j_r) \prod_{i=1}^{r} \left\{ \frac{[(-u_i)^{m_i} y_i]^{j_i}}{j!!} \right\} \cdot$$

$$\cdot \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{u_1^{n_1}}{n_1!} \cdots\cdots \frac{u_r^{n_r}}{n_r!} \sum_{j=0}^{m+N+J} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \cdot \binom{j}{l}$$

$$\cdot (\xi)_l \left(\frac{hl+\alpha}{k}\right)_{m+N+J}$$

Now we appeal to the well known series identity.

$$(4.3.5)\quad \sum_{n_1,\dots,n_r=0}^{\infty} f(n_1+\cdots+n_r)\frac{u_1^{n_1}}{n_1!}\cdots\frac{u_r^{n_r}}{n_r!} = \sum_{n=0}^{\infty} f(n)\frac{(u_1+\cdots+u_r)^n}{n!},$$

the equation (4.3.3) becomes

$$(4.3.6)\quad \Omega(u_1,\dots,u_r) = k^m \sum_{n,j_1,\dots,j_r=0}^{\infty} \frac{(u_1+\cdots+u_r)^n}{n!}\Lambda(j_1,\dots,j_r)$$

$$\cdot\prod_{i=1}^{r}\left\{\frac{[(-u_i)^{m_i}y_i]^{j_i}}{j_i!}\right\}\sum_{j=0}^{m+n+J}\frac{(px^h)^j}{j!}\sum_{l=0}^{j}(-1)^l\binom{j}{l}(\xi)_l\left(\frac{hl+\alpha}{k}\right)_{m+n+J},$$

Where $J$ is defined as before by (4.2.1), the inner most sum in (4.3.5) is the $j^{th}$ difference of a polynomial of degree $m+n+J$ in $\alpha$; it is nill when $j > m+n+J$.

Hence

$$\sum_{j=0}^{m+n+J}\frac{(px^h)^j}{j!}\sum_{l=0}^{j}(-1)^l\binom{j}{l}(\xi)_l\left(\frac{hl+\alpha}{k}\right)_{m+n+J}$$
$$= \sum_{l=0}^{\infty}\left(\frac{hl+\alpha}{k}\right)_{m+n+J}\frac{(-px^h)^l}{l!}(\xi)_l\sum_{j=0}^{\infty}\frac{(px^h)^j}{j!}$$
$$= \exp(px^h)\sum_{l=0}^{\infty}\left(\frac{hl+\alpha}{k}\right)_{m+n+J}\frac{(-px^h)^l}{l!}(\xi)_l.$$

Substituting this in equation (4.3.5) and applying the binomial expansion to sum the resulting n - series we have

$$(4.3.7)\quad \Omega(u_1,\dots,u_r) = k^m\sum_{n,j_1,\dots,j_r=0}^{\infty}\frac{(u_1+\cdots+u_r)^n}{n!}\Lambda(j_1,\dots,j_r)$$

$$\cdot\prod_{i=1}^{r}\left\{\frac{[(-u_i)^{m_i}y_i]^{j_i}}{j_i!}\right\}\exp(px^h)\sum_{l=0}^{\infty}\left(\frac{hl+\alpha}{k}\right)_{m+n+J}$$

$$\cdot\frac{(-px^h)^l}{l!}(\xi)_l.$$

<u>Case I:</u> When r = 1,

$$(4.3.8)\quad \Omega(U_1) = k^m \exp(px^h) \sum_{n, j_1=0}^{\infty} \frac{U_1^n}{n!} \Lambda(j_1) \left\{ \frac{[(-U_1)^{m_1} y_1]^{j_1}}{j_1!} \right\}$$

$$\cdot \sum_{l=0}^{\infty} \left(\frac{hl+\alpha}{k}\right)_{m+n+m_1 j_1} \frac{(-px^h)^l}{l!} (\xi)_l$$

$$= k^m \exp(px^h) \sum_{l=0}^{\infty} \sum_{j_1=0}^{\infty} \Lambda(j_1) \left(\frac{hl+\alpha}{k}\right)_{m+m_1 j_1} \frac{(-px^h)^l}{l!}$$

$$\cdot (\xi)_l \left\{ \frac{[(-U_1)^{m_1} y_1]^{j_1}}{j_1!} \right\} \sum_{n=0}^{\infty} \frac{U_1^n}{n!} \left(\frac{hl+\alpha}{k} + m + m_1 j_1\right)_n$$

$$= k^m \exp(px^h) \sum_{l, j_1=0}^{\infty} \Lambda(j_1) \left\{ \frac{[(-U_1)^{m_1} y_1]^{j_1}}{j_1!} \right\} \frac{(-px^h)^l}{l!}$$

$$\cdot (\xi)_l \left(\frac{hl+\alpha}{k}\right)_{m+m_1 j_1} (1-U_1)^{-\left(\frac{hl+\alpha}{k}\right) - m - m_1 j_1}$$

$$= k^m \exp(px^h) \sum_{l, j_1=0}^{\infty} \Lambda(j_1) \left\{ \frac{[(-U_1/\Delta_1)^{m_1} y_1]^{j_1}}{j_1!} \right\} \frac{(\xi)_l}{l!} \cdot$$

$$\cdot \left(\frac{-px^h}{\Delta_1^{h/k}}\right)^l \cdot \Delta_1^{-m-\alpha/k} \left(\frac{hl+\alpha}{k}\right)_{m+m_1 j_1} .$$

<u>Case II:</u> When r = 2,

$$\Omega(U_1, U_2) = k^m \exp(px^h) \sum_{n=0}^{\infty} \sum_{j_1, j_2=0}^{\infty} \frac{(U_1+U_2)^n}{n!} \Lambda(j_1, j_2)$$

$$\cdot \prod_{i=1}^{2} \left\{ \frac{[(-U_i)^{m_i} y_i]^{j_i}}{j_i} \right\} \sum_{l=0}^{\infty} \left(\frac{hl+\alpha}{k}\right)_{m+m_1 j_1 + m_2 j_2} (-px^h)^l \frac{(\xi)_l}{l!}$$

$$= k^m \exp(px^h) \sum_{l,j_1,j_2=0}^{\infty} \Lambda(j_1,j_2) \left(\frac{hl+\alpha}{k}\right)_{m+m_1j_1+m_2j_2} (-px^h)^l$$

$$\cdot \frac{(\xi)_l}{l!} \prod_{i=1}^{2} \left\{ \frac{[(-v_i)^{m_i} y_i]^{j_i}}{j_i!} \right\} \sum_{n=0}^{\infty} \frac{(v_1+v_2)^n}{n!} \left(\frac{hl+\alpha}{k} + m + m_1j_1 + m_2j_2\right)$$

$$= k^m \exp(px^h) \sum_{l,j_1,j_2=0}^{\infty} \Lambda(j_1,j_2) \prod_{i=1}^{2} \left\{ \frac{[(-v_i)^{m_i} y_i]^{j_i}}{j_i!} \right\} \frac{(\xi)_l}{l!}$$

$$\cdot (-px^h)^l \left(\frac{hl+\alpha}{k}\right)_{m+m_1j_1+m_2j_2} (1-(v_1+v_2))^{-\left(\frac{hl+\alpha}{k}\right)-m-m_1j_1-m_2j_2}$$

$$= k^m \exp(px^h) \sum_{l,j_1,j_2=0}^{\infty} \Lambda(j_1,j_2) \prod_{i=1}^{2} \left\{ \frac{[(-v_i/\Delta_i)^{m_i} y_i]^{j_i}}{j_i!} \right\} \frac{(\xi)_l}{l!}$$

$$\cdot \left( \frac{-px^h}{\Delta_2^{h/k}} \right)^l \Delta_2^{-m-\alpha/k} \left(\frac{hl+\alpha}{k}\right)_{m+m_1j_1+m_2j_2} ,$$

and so in general, we finally obtain

$$(4.3.9) \quad \Omega(v_1,\dots,v_r) = k^m \exp(px^h)\, \Delta_r^{-m-\frac{\alpha}{k}} \sum_{l,j_1,\dots,j_r=0}^{\infty} \left(\frac{hl+\alpha}{k}\right)_{m+m_1j_1+\dots+m_rj_r}$$

$$\cdot \frac{(\xi)_l}{l!} \Lambda(j_1,\dots,j_r) \left(\frac{-px^h}{\Delta_r^{h/k}}\right)^l \prod_{i=1}^{r} \left\{ \frac{[(-v_i/\Delta_r)^{m_i} y_i]^{j_i}}{j_i!} \right\}$$

or $\displaystyle\sum_{n_1,\dots,n_r=0}^{\infty} (m+n_1+\dots+n_r)\, F^{(\alpha)}_{m+n_1+\dots+n_r}(x,h,p,k)$

$$\cdot H[n_1,\dots,n_r;\, y_1,\dots,y_r] \frac{(v_1/k)^{n_1}}{n_1!} \cdots \frac{(v_r/k)^{n_r}}{n_r!}$$

$$= k^m \exp(px^h)\, \Delta_r^{-m-\alpha/k} \sum_{n,n_1,\dots,n_r=0}^{\infty} \left(\frac{hn+\alpha}{k}\right)_{m+m_1n_1+\dots+m_rn_r}$$

$$\cdot \frac{(\xi)_n}{n!} \Delta(n_1,\dots,n_r) \left(-\frac{px^h}{\Delta_r^{h/k}}\right)^n \prod_{i=1}^{r} \left\{ \frac{[(-v_i/\Delta_r)^{m_i} y_i]^{n_i}}{n_i!} \right\} .$$

This completes the proof of (4.3.2)

## 4.4 APPLICATIONS:

By assigning suitable special values to the arbitrary coefficients $\Lambda(j_1,\ldots,j_r)$, the multiple sum in (4.1.1) can indeed be expressed in terms of the generalized Lauricella hypergeometric function of r variables. ( See Appendix ).

$$(4.4.1)\quad \sum_{n_1,\ldots,n_r=0}^{\infty} (m+n_1+\cdots+n_r)!\, G^{(\alpha)}_{m+n_1+\cdots+n_r}(x,h,p,k)$$

$$\cdot F^{A:\,1+B';\,\ldots;\,1+B^{(r)}}_{C:\,D';\,\ldots;\,D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[-n_1:m_1],[(b'):\phi'];\ldots;[-n_r:m_r],[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; y_1,\ldots,y_r\right)\left(\frac{u_1}{k}\right)^{n_1}\cdots\left(\frac{u_r}{k}\right)^{n_r}$$

$$= k^m\left(\frac{\alpha}{k}\right)_m \exp(px^h)\,\Delta_r^{-m-\alpha/k}\, F^{1+A:\,0;\,B';\,\ldots;\,B^{(r)}}_{C:\,1;\,D';\,\ldots;\,D^{(r)}}$$

$$\left(\begin{matrix}[m+\alpha/k:h/k,m_1,\ldots,m_r],[(a):0,\theta',\ldots,\theta^{(r)}]:\text{—};\;[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):0,\psi',\ldots,\psi^{(r)}]:[\alpha/k:h/k];\;[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\;\Xi_0,\Xi_1,\ldots,\Xi_r\right)$$

where $h/k>0$, $\Delta_r$ is given by (4.1.2) and

$$(4.4.2)\quad \Xi_0 = -\frac{px^h}{\Delta_r^{h/k}},\quad \Xi_i = y_i\left(-\frac{u_i}{\Delta_r}\right)^{m_i},\quad i=1,\ldots,r.$$

Now substituting A = C = 0 in (4.4.1) and for convenience let each of the positive coefficients $\phi_j^{(i)}$, $j = 1, \ldots, B^{(i)}$; $\delta_j^{(i)}$, $j = 1, \ldots, D^{(i)}$ $(i = 1, \ldots, r)$ equal 1. Denoting the array of parameters

$$(-n_i + j - 1)/m_i \quad, j = 1, \ldots, m_i ,$$

by $\Delta(m_i; -n_i)$, $i = 1, \ldots, r$, we thus find from (4.4.1) that the equation becomes

$$(4.4.3) \quad \sum_{n_1, \ldots, n_r = 0}^{\infty} (m + n_1 + \cdots + n_r)! \, G^{(\alpha)}_{m+n_1+\cdots+n_r}(x, h, p, k)$$

$$\cdot \prod_{i=1}^{r} \left\{ \underset{m_i + B^{(i)}}{} F_{D^{(i)}} \left[ \begin{matrix} \Delta(m_i; -n_i), (b^{(i)}); \\ (d^{(i)}); \end{matrix} \; y_i \, m_i^{m_i} \right] \left( \frac{u_1}{k} \right)^{n_i} \right\}$$

$$= k^m \left( \frac{\alpha}{k} \right)_m \exp(px^h) \, \Delta_r^{-m-\alpha/k}$$

$$\cdot F^{1:0:B';\ldots;B^{(r)}}_{0:1:D';\ldots;D^{(r)}} \left( \begin{matrix} [m + \alpha/k : h/k, m_1, \ldots, m_r] : \text{---}; \\ \text{------------} : [\alpha/k : h/k]; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):1]; \ldots; [(b^{(r)}):1]; \\ [(d'):1]; \ldots; [(d^{(r)}):1]; \end{matrix} \; \Xi_0, \Xi_1, \ldots, \Xi_r \right),$$

where $h/k > 0$ , $\Delta_r$ is given by (4.1.2), and $\Xi_0, \Xi_1, \ldots \Xi_r$ are defined by (4.4.2).

<u>Proof of (4.4.3):</u>

Consider

$$\sum_{n_1,\ldots,n_r=0}^{\infty} (m+n_1+\cdots+n_r)!\, G^{(\alpha)}_{m+n_1+\cdots+n_r}(x,h,p,k)$$

$$F^{1+B':\cdots\cdots;1+B^{(r)}}_{D';\cdots\cdots;D^{(r)}}\left(\begin{matrix}[-n_1:m_1],[(b'):1]\cdots\cdots[-n_r:m_r],[(b^{(r)}):1];\\ [(d'):1]\cdots\cdots[(d^{(r)}):1];\end{matrix}\; y_1,\ldots,y_r\right)\left(\frac{u_1}{k}\right)^{n_1}\cdots\left(\frac{u_r}{k}\right)^{n_r}$$

$$= k^m\left(\frac{\alpha}{k}\right)_m \exp(px^h)\,\Delta_r^{-m-\alpha/k}$$

$$F^{1:0:B';\cdots;B^{(r)}}_{0:1:D';\cdots;D^{(r)}}\left(\begin{matrix}[m+\alpha/k;h/k,m_1,\ldots,m_r]; \text{---}\\ \text{---};[\alpha/k;h/k]\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):1];\cdots\cdots;[(b^{(r)}):1];\\ [(d'):1];\cdots\cdots;[(d^{(r)}):1];\end{matrix}\; -\frac{px^h}{\Delta_r^{h/k}}, y_1\left(-\frac{u_1}{\Delta_r}\right)^{m_1}\cdots y_r\right)$$

Case I : When r = 1

$$\sum_{n_1=0}^{\infty}(m+n_1)!\, G^{(\alpha)}_{m+n_1}(x,h,p,k)\, F^{1+B'}_{D'}\left(\begin{matrix}(-n_1)_{m_1}, & (b'); \\ & (d');\end{matrix}\; y_1\right)\left(\frac{u_1}{k}\right)^{n_1}$$

$$= k^m\left(\frac{\alpha}{k}\right)_m \exp(px^h)\,\Delta_1^{-m-\alpha/k}$$

$$F^{1:0:B'}_{0:1:D'}\left(\begin{matrix}[m+\frac{\alpha}{k};h/k,m_1]: \text{---} & ;[(b'):1]; \\ \text{---};[\alpha/k;h/k]; & [(d'):1];\end{matrix}\; -\frac{px^h}{\Delta_1^{h/k}}, y_1\left(-\frac{u_1}{\Delta_1}\right)^{m_1}\right)$$

$$\sum_{n_1=0}^{\infty} (m+n_1)!\, G^{(\alpha)}_{m+n_1}(x,h,p,k)\, F^{1+B'}_{D'}\left(\left(\frac{-n_1+j-1}{m_1}\right), (b') \,;\; \underset{(d')\,;}{} y_1 m_1^{m_1}\right)\left(\frac{u_1}{k}\right)^{n_1}$$

$$= k^m \left(\frac{\alpha}{k}\right)_m \exp(px^h)\, \Delta_1^{-m-\alpha/k} \,.$$

$$F^{1:0:B'}_{0:1:D'}\left(\begin{matrix}[m+\frac{\alpha}{k};\frac{h}{k},m_1];\text{———} \;;[(b'):1];\\ \text{—————} ;[\frac{\alpha}{k};h/k];[(d'):1];\end{matrix}\; \frac{-px^h}{\Delta_1^{h/k}k}, y_1\left(\frac{-u_1}{\Delta_1}\right)\right)$$

$$\sum_{n_1=0}^{\infty} (m+n_1)!\, G^{(\alpha)}_{m+n_1}(x,h,p,k)\left\{{}_{m_1+B'}F_{D'}\left(\begin{matrix}\Delta(m_1;-n_1), (b')\,; \\ (d')\,;\end{matrix}\; y_1 m_1^{m_1}\right)\left(\frac{u_1}{k}\right)^{n_1}\right.$$

$$= k^m \left(\frac{\alpha}{k}\right) \exp(px^h)\cdot \Delta_1^{-m-\frac{\alpha}{k}} \,.$$

$$F^{1:0:B'}_{0:1:D'}\left(\begin{matrix}[m+\frac{\alpha}{k};\frac{h}{k},m_1];\text{——} \;;[(b'):1];\\ \text{—————};[\frac{\alpha}{k};h/k];[(d'):1];\end{matrix}\; \frac{-px^h}{\Delta_1^{h/k}k}, y_1\left[\frac{-u_1}{\Delta_1}\right]^{m_1}\right) \cdot$$

Case II : When r = 2.

$$\sum_{n_1,n_2=0}^{\infty} (m+n_1+n_2)!\, G^{(\alpha)}_{m+n_1+n_2}(x,h,p,k)\, F^{1+B^{1,2}}_{D^{1,2}}\left(\left(\frac{-n_{1,2}+j-1}{m_{1,2}}\right), \begin{matrix}(b^{1,2})\,;\\ (d^{1,2})\,;\end{matrix}\; y_{1,2} m_{1,2}^{m_{1,2}}\right)\left(\frac{u_1}{k}\right)$$

$$= k^m \left(\frac{\alpha}{k}\right)_m \exp(px^h)\, \Delta_{1,2}^{-m-\alpha/k}$$

$$\cdot F^{1:0:B^{1,2}}_{0:1:D^{1,2}}\left(\begin{matrix}[m+\frac{\alpha}{k};\frac{h}{k},m_{1,2}],\text{- - -};[b^1:1],[b^2:1];\\ \text{—————},[\alpha/k;\frac{h}{k}];[d^1:1],[d^2:1];\end{matrix}\; \frac{-px^h}{\Delta_{1,2}^{h/k}k}, y_{1,2}\left(\frac{-u_{1,2}}{\Delta_{1,2}}\right)^{m_{1,2}}\right)$$

or
$$\sum_{n_1,n_2=0}^{\infty} (m+n_1+n_2)!\, G^{(\alpha)}_{m+n_1+n_2}(x,h,p,k)\left\{{}_{m_{1,2}+B^{1,2}}F_{D^{1,2}}\left(\begin{matrix}\Delta(m_{1,2};-n_{1,2}),(b)^{1,2},\\ (d)^{1,2};\end{matrix}\; y_{1,2}m\right) \cdot \left(\frac{u_{1,2}}{k}\right)^{n_{1,2}}\right\}$$

$$= k^m \left(\frac{\alpha}{k}\right)_m \exp(px^h)\, \Delta_{1,2}^{-m-\frac{\alpha}{k}}$$

$$F^{1:0:B^{1,2}}_{0:1:D^{1,2}}\left(\begin{matrix}[m+\alpha/k;h/k,m_{1,2}];\text{——} \;;[b^1:1],[b^2:1];\\ \text{—————};[\alpha/k;h/k];[d^1:1],[d^2:1];\end{matrix}\; \frac{-px^h}{\Delta_{1,2}^{h/k}k}, y_{1,2}\left(\frac{-u_{1,2}}{\Delta_{1,2}}\right)^{m_{1,2}}\right)$$

and repeating this process upto $\gamma$ times, we get

$$\sum_{n_1,\dots,n_\gamma=0}^{\infty} (m+n_1+\cdots+n_\gamma)!\, G^{(\alpha)}_{m+n_1+\cdots+n_\gamma}(x,t,\rho,k)$$

$$\cdot\prod_{i=1}^{\gamma}\left\{ {}_{m_i+B^{(i)}}F_{D^{(i)}}\left[\begin{matrix}\Delta(m_i;-n_i),(b^{(i)});\\ (d^{(i)});\end{matrix}\; y_i m_i^{m_i}\right]\left(\frac{u_i}{k}\right)^{n_i}\right\}$$

$$= k^m\left(\frac{\alpha}{k}\right)_m \exp(px^\rho)\,\Delta_\gamma^{-m-\alpha/k}$$

$$\cdot F^{1:0:B';\dots;B^{(\gamma)}}_{0:1:D';\dots;D^{(\gamma)}}\left(\begin{matrix}[m+\frac{\alpha}{k};\rho/k;\ m_1,\dots,m_\gamma]:\text{———};\\ \text{———}\ ;[\frac{\alpha}{k}:\frac{\rho}{k}];\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):1];\dots;[(b^{(\gamma)}):1];\\ [(d'):1];\dots;[(d^{(\gamma)}):1];\end{matrix}\ \Xi_0,\Xi_1,\dots,\Xi_\gamma\right),$$

where $\Delta_\gamma$ is given by (4.1.2) and $\Xi_0, \Xi_1, \dots, \Xi_\gamma$ are defined by (4.4.2), which proves (4.4.3).

Now substituting the values $h = p = 1$, $A=B^{(i)}=C=D^{(i)}-1=0$, $d_1^{(i)}=1+\beta_i$, $s_1^{(i)}=s_i$, $m_i=1$ and $\alpha=\alpha+1$, $y_i=y_i s^i$, $i=1,\dots,\gamma$ in equation (4.4.1) and by use of relation (1.4.22) and (3.1.1), we have a multilinear generating function for Konhauser biorthogonal polynomials:

$$\text{(4.4.4)}\qquad \sum_{n_1,\dots,n_\gamma=0}^{\infty} (m+n_1+\cdots+n_\gamma)!\, Y^{\alpha}_{m+n_1+\cdots+n_\gamma}(x;k)$$

$$\cdot\prod_{i=1}^{\gamma}\left\{ Z^{\beta_i}_{n_i}(y_i;s_i)\frac{u_i^{n_i}}{(1+\beta_i)_{s_i n_i}}\right\}$$

$$= \left(\frac{\alpha+1}{k}\right)_m e^x\,\Delta_\gamma^{-m-(\alpha+1)/k}$$

$$\cdot F^{1:0;\,\ldots\,;0}_{0:1;\,\ldots\,;1}\left(\begin{array}{l}[m+(\alpha+1)/k : 1/k,\ 1,\ldots,1] : \text{———} ; \text{———} ; \ldots ; \text{———} ; \\ \text{————————} : [(\alpha+1)/k : 1/k] ; [1+\beta_1 : s_1] ; \ldots ; [1+\beta_r : s_r] ;\end{array} -\frac{x}{\Delta_r^{1/k}},\ -\frac{u_1 y_1^{s_1}}{\Delta_r},\ \ldots,\ -\frac{u_r y_r^{s_r}}{\Delta_r}\right),$$

where $\alpha > -1$ ; $\beta_i > -1$ ; $k, s_i = 1, 2, 3, \ldots$ $\forall\, i \in \{1, \ldots, r\}$.

## APPENDIX

1. **Lagrange's theorem:** If $\phi(z)$ is holomorphic at $z = x$ and $\phi(x) \neq 0$, and if $z = x + t\phi(z)$ then

(1) $$f(z) = f(x) + \sum_{n=1}^{\infty} \frac{t^n}{n!} D^{n-1}\left[\{\phi(x)\}^n f(x)\right].$$

This expansion in modified form, can be expressed as

(2) $$\frac{F(z)}{1-t\phi'(z)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n\left[\{\phi(x)\}^n F(x)\right].$$

For details, see Whittakar and Watson [37].

2. **Extended Carlitz theorem:**

Let $A(z)$, $B(z)$ and $z^{-1}C(z)$ be arbitrary function which are analytic in a neighbourhood of the origin and assume that

(3) $$A(0) = B(0) = C'(0) = 1.$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}$ by means of

(4) $$A(z)[B(z)]^{\alpha} \exp(x\,C(z)) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x)\frac{z^n}{n!},$$

where $\alpha$ and $x$ are arbitrary complex numbers, independent of z. Then, for arbitrary parameters $\lambda$ and $y$ independent of z,

(5) $$\sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)\frac{t^n}{n!} = \frac{A(\xi)[B(\xi)]^{\alpha}\exp(x\,C(\xi))}{1-\xi\{\lambda[B'(\xi)/B(\xi)] + yC'(\xi)\}},$$

where $\xi = t[B(\xi)]^{\lambda}\exp(y\,C(\xi))$.

For proof of this theorem, see Srivastava and Manocha [35, p.37]

**3** Generalised Hypergeometric Function:

Define

(6) $${}_pF_q[\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z]$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_p)_n}{n! (\beta_1)_n, \ldots, (\beta_q)_n} z^n ; = {}_pF_q \begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_p ; & \\ & z \\ \beta_1, \beta_2, \ldots, \beta_q ; & \end{bmatrix},$$

where $(\lambda)_n$ is Pachchamer symbol defined by

$$(\lambda)_n = \lambda(\lambda+1) \ldots (\lambda+n-1), \; n \geq 1,$$

$$(\lambda)_0 = 1, \quad \lambda \neq 0 .$$

The function defined by (6) is known as generalised hypergeometric series or function of z, and parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$; $\beta_1, \beta_2, \ldots, \beta_q$ take the values in such a way that the infinite series either terminates or is convergent. In particular, denominator parameters, $\beta$'s are neither Zero nor a negative integer.

We can easily see that

(7) $$\frac{d}{dz} {}_pF_q[\alpha_1, \ldots, \alpha_p ; \beta_1, \ldots, \beta_q ; z]$$

$$= \frac{\alpha_1, \alpha_2, \ldots, \alpha_p}{\beta_1, \beta_2, \ldots, \beta_q} \, {}_pF_q[\alpha_1+1, \ldots, \alpha_p+1 ; \beta_1+1, \ldots, \beta_q+1 ; z] .$$

For more details one can refer to Rainville [25].

4. Generalised Lauricella Hypergeometric Function of many variables:

The generalised Lauricella hypergeometric function of n variables has been denoted and defined as follows [35]:

$$F_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$= F_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}}\left(\begin{matrix} [(a);\theta^{(1)},\ldots,\theta^{(n)}]:[(b^{(1)});\phi^{(1)}],\ldots,[(b^{(n)});\phi^{(n)}]; \\ [(c);\psi^{(1)},\ldots;\psi^{(n)}]:[(d^{(1)});\delta^{(1)}],\ldots,[(d^{(n)});\delta^{(n)}]; \end{matrix}\; z_1, z_2, \ldots, z_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \Omega(m_1, m_2, \ldots, m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},$$

where

$\Omega(m_1, m_2, \ldots, m_n)$

$$= \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+m_2\phi_j^{(2)}+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

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# SOME BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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Reprinted from Pacific Journal of Mathematics Vol. 98, No. 1
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PACIFIC JOURNAL OF MATHEMATICS
Vol. 98, No. 1, 1982

# SOME BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

H. M. SRIVASTAVA

**Joseph D. E. Konhauser discussed two polynomial sets $\{Y_n^\alpha(x;k)\}$ and $\{Z_n^\alpha(x;k)\}$, which are biorthogonal with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0,\infty)$, where $\alpha > -1$ and $k$ is a positive integer. The present paper attempts at exploring certain novel approaches to these biorthogonal polynomials in simple derivations of their several interesting properties. Many of the results obtained here are believed to be new; others were proven in the literature by employing markedly different techniques.**

1. Introduction. Konhauser ([10]; see also [9]) has considered two classes of polynomials $Y_n^\alpha(x;k)$ and $Z_n^\alpha(x;k)$, where $Y_n^\alpha(x;k)$ is a polynomial in $x$, while $Z_n^\alpha(x;k)$ is a polynomial in $x^k$, $\alpha > -1$ and $k = 1, 2, 3, \cdots$. For $k = 1$, these polynomials reduce to the Laguerre polynomials $L_n^{(\alpha)}(x)$, and their special cases when $k = 2$ were encountered earlier by Spencer and Fano [19] in certain calculations involving the penetration of gamma rays through matter, and were subsequently discussed by Preiser [16]. Furthermore, we have [10, p. 303]

$$\int_0^\infty x^\alpha e^{-x} Y_i^\alpha(x;k) Z_j^\alpha(x;k)\,dx = \frac{\Gamma(kj+\alpha+1)}{j!}\delta_{ij}\,, \quad \forall\, i, j \in \{0, 1, 2, \cdots\}\,, \tag{1.1}$$

which exhibits the fact that the polynomial sets $\{Y_n^\alpha(x;k)\}$ and $\{Z_n^\alpha(x;k)\}$ are *biorthogonal* with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0,\infty)$, it being understood that $\alpha > -1$, $k$ is a positive integer, and $\delta_{ij}$ is the Kronecker delta.

An explicit expression for the polynomials $Z_n^\alpha(x;k)$ was given by Konhauser in the form [10, p. 304, Eq. (5)]

$$Z_n^\alpha(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)}\,. \tag{1.2}$$

As for the polynomials $Y_n^\alpha(x;k)$, Carlitz [3] subsequently showed that [*op. cit.*, p. 427, Eq. (9)]

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n\,, \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(1.4)\qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1\,, & \text{if} \quad n=0,\ \lambda \neq 0\,, \\ \lambda(\lambda+1)\cdots(\lambda+n-1)\,, & \forall\, n \in \{1, 2, 3, \cdots\}\,. \end{cases}$$

The object of the present paper is to show that several interesting properties of the biorthogonal polynomials $Y_n^\alpha(x;k)$ and $Z_n^\alpha(x;k)$ follow fairly readily from relatively more familiar results by applying the explicit expressions (1.2) and (1.3). A number of properties thus derived are believed to be new, and others were proven in the literature by employing markedly different techniques.

2. The biorthogonal polynomials $Y_n^\alpha(x;k)$. We begin by recalling the polynomials $G_n^{(\alpha)}(x, r, p, k)$ which were introduced by Srivastava and Singhal [24] in an attempt to provide an elegant unification of the various known generalizations of the classical Hermite and Laguerre polynomials. These polynomials are defined by the generalized Rodrigues formula [*op. cit.*, p. 75, Eq. (1.3)]

$$(2.1)\qquad G_n^{(\alpha)}(x, r, p, k) = \frac{x^{-kn-\alpha}\exp(px^r)}{n!}(x^{k+1}D_x)^n\{x^\alpha \exp(-px^r)\}\,,$$

where $D_x = d/dx$, and the parameters $\alpha$, $k$, $p$ and $r$ are unrestricted, in general. We also have the explicit polynomial expression [24, p. 77, Eq. (2.1)]

$$(2.2)\qquad G_n^{(\alpha)}(x, r, p, k) = \frac{k^n}{n!}\sum_{i=0}^{n}\frac{(px^r)^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{rj+\alpha}{k}\right)_n\,.$$

On comparing (2.2) with Carlitz's result (1.3), we at once get the known relationship [23, p. 315, Eq. (83)]

$$(2.3)\qquad Y_n^\alpha(x;k) = k^{-n}\,G_n^{(\alpha+1)}(x, 1, 1, k)\,,\quad \alpha > -1\,,\quad k = 1, 2, 3, \cdots,$$

which evidently enables us to derive the following properties of the Konhauser biorthogonal polynomials $Y_n^\alpha(x;k)$ by suitably specializing those of the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, r, p, k)$.

I. *Rodrigues' formula.* In (2.1) we set $p = r = 1$, replace $\alpha$ by $\alpha + 1$, and appeal to the relationship (2.3). We thus obtain

$$(2.4)\qquad Y_n^\alpha(x;k) = \frac{x^{-kn-\alpha-1}e^x}{k^n n!}(x^{k+1}D_x)^n\{x^{\alpha+1}e^{-x}\}\,,$$

where, by definition, $\alpha > -1$ and $k$ is now restricted to be a positive integer.

Alternatively, we may recall that [15, p. 802, Eq. (2.6)]

$$(2.5)\qquad Y_n^\alpha(x;k) = \frac{x^{k-\alpha-1}e^x}{n!} D_s^n\{s^{n-1+(\alpha+1)/k}\exp(-s^{1/k})\}\Big|_{s=x^k},$$

which indeed is equivalent to

$$(2.6)\qquad Y_n^\alpha(x;k) = \frac{x^{k-\alpha-1}e^x}{n!}[s^{-n-1}(s^2D_s)^n\{s^{(\alpha+1)/k}\exp(-s^{1/k})\}]_{s=x^k},$$

since

$$(2.7)\qquad (x^2D_x)^n\{g(x)\} = x^{n+1}D_x^n\{x^{n-1}g(x)\}$$

for every non-negative integer $n$.

Now we set $s = x^k$ and $s^2D_s = k^{-1}x^{k+1}D_x$ in (2.6), and the Rodrigues formula (2.4) follows at once.

Incidentally, the Rodrigues type representation (2.4) is due to Calvez *et* Génin [2, p. A41, Eq. (1)]; it is stated *slightly differently* in a recent paper by Patil and Thakare [12, p. 921, Eq. (1.2)].

II. *Recurrence relations.* In view of the relationship (2.3), the known results [24, p. 80, Eq. (4.3), (4.4), (4.5) and (4.6)] readily yield

$$(2.8)\qquad k(n+1)Y_{n+1}^\alpha(x;k) = xD_xY_n^\alpha(x;k) + (kn+\alpha-x+1)Y_n^\alpha(x;k),$$

$$(2.9)\qquad D_xY_n^\alpha(x;k) = Y_n^\alpha(x;k) - Y_n^{\alpha+1}(x;k),$$

$$(2.10)\qquad (\alpha-k+1)Y_n^\alpha(x;k) = xY_n^{\alpha+1}(x;k) + (n+1)kY_{n+1}^{\alpha-k}(x;k)$$

and

$$(2.11)\qquad k(n+1)Y_{n+1}^\alpha(x;k) = (kn+\alpha+1)Y_n^\alpha(x;k) - xY_n^{\alpha+1}(x;k).$$

The recurrence relation (2.8) was given earlier by Konhauser [10, p. 308, Eq. (16)], while (2.9), (2.10) and (2.11) are believed to be new. Notice, however, that by eliminating the term $xY_n^{\alpha+1}(x;k)$ between (2.10) and (2.11) we obtain

$$(2.12)\qquad Y_{n+1}^{\alpha-k}(x;k) = Y_{n+1}^\alpha(x;k) - Y_n^\alpha(x;k),$$

which is equivalent to the familiar generalization (*cf.* [10], p. 311) of a well-known recurrence relation for the Laguerre polynomials [18, p. 203, Eq. (8)].

III. *Operational formulas.* Making use of the relationship (2.3), we can specialize the Srivastava-Singhal results [24, p. 85, Eq. (7.5) and (7.6)] to obtain the following operational formulas involving the biorthogonal polynomials $Y_n^\alpha(x;k)$:

$$(2.13)\qquad \prod_{j=0}^{n-1}(\delta+\alpha+jk-x+1) = k^n n!\sum_{j=0}^{n}\frac{(kx^k)^{-j}}{j!}Y_{n-j}^\alpha(x;k)(x^{k+1}D_x)^j.$$

and

$$Y_n^\alpha(x; k) = \frac{1}{k^n n!} \prod_{j=0}^{n-1} (\delta + \alpha + jk - x + 1)\cdot 1 \,, \tag{2.14}$$

where $\delta = xD_x$.

IV. *Generating functions.* From the known results [24, p. 78, Eq. (3.2); p. 79, Eq. (3.4) and (3.6)], due to Srivastava and Singhal [24], it readily follows on appealing to (2.3) that

$$\sum_{n=0}^{\infty} Y_n^\alpha(x; k)t^n = (1-t)^{-(\alpha+1)/k} \exp(x[1-(1-t)^{-1/k}]) \,, \tag{2.15}$$

$$\sum_{n=0}^{\infty} Y_n^{\alpha-kn}(x; k)t^n = (1+t)^{(\alpha-k+1)/k} \exp(x[1-(1+t)^{1/k}]) \,, \tag{2.16}$$

and

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^\alpha(x; k)t^n = (1-t)^{-m-(\alpha+1)/k} \exp(x[1-(1-t)^{-1/k}]) Y_m^\alpha(x(1-t)^{-1/k}; k) \,, \tag{2.17}$$

where $m$ is a non-negative integer.

Furthermore, by using the definition (2.1) and the aforementioned result [24, p. 79, Eq. (3.4)], it is not difficult to derive the generating function

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{m+n}{n} G_{m+n}^{(\alpha-kn)}(x, r, p, k)t^n &= (1+kt)^{(\alpha-k)/k} \exp(px^r[1-(1+kt)^{r/k}]) \\ &\times G_m^{(\alpha)}(x(1+kt)^{1/k}, r, p, k) \,, \quad k \neq 0 \,,\end{aligned} \tag{2.18}$$

which, for $p = r = 1$, yields a generalization of (2.16) in the form:

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha-kn}(x; k)t^n &= (1+t)^{(\alpha-k+1)/k} \exp(x[1-(1+t)^{1/k}]) \\ &\times Y_m^\alpha(x(1+t)^{1/k}; k) \,, \quad \forall\, m \in \{0, 1, 2, \cdots\} \,,\end{aligned} \tag{2.19}$$

where, by definition, $k$ is a positive integer.

The generating function (2.15) was derived earlier by Carlitz [3, p. 426, Eq. (8)], while (2.16), (2.17) and (2.19) are due to Calvez *et* Génin [2]. In fact, (2.15) and (2.17) were also given *independently* by Prabhakar [15, p. 801, Eq. (2.3); p. 803, Eq. (3.3)].

Incidentally, in view of the known generating function [24, p. 78, Eq. (3.2)] and Lagrange's expansion in the form [13, p. 146,

Problem 207]:

$$\frac{f(\zeta)}{1 - t\phi'(\zeta)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_x^n\{f(x)[\phi(x)]^n\}\Big|_{x=0}, \tag{2.20}$$

where

$$\zeta = t\,\phi(\zeta), \qquad \phi(0) \neq 0, \tag{2.21}$$

it is fairly easy to show that

$$\begin{aligned} &\sum_{n=0}^{\infty} G_n^{(\alpha+\beta n)}([x^r + ny^r]^{1/r}, r, p, k)t^n \\ &\qquad = \frac{(1-u)^{-\alpha/k}\exp(px^r[1-(1-u)^{-r/k}])}{1 - k^{-1}u(1-u)^{-1}[\beta - rpy^r(1-u)^{-r/k}]}, \quad k \neq 0, \end{aligned} \tag{2.22}$$

or, equivalently,

$$\begin{aligned} &\sum_{n=0}^{\infty} G_n^{(\alpha+\beta n)}([x^r + ny^r]^{1/r}, r, p, k)t^n \\ &\qquad = \frac{(1+v)^{\alpha/k}\exp(px^r[1-(1+v)^{r/k}])}{1 - k^{-1}v[\beta - rpy^r(1+v)^{r/k}]}, \quad k \neq 0, \end{aligned} \tag{2.23}$$

where $u$ and $v$ are functions of $t$ defined *implicitly* by

$$u = kt(1-u)^{-\beta/k}\exp(py^r[1-(1-u)^{-r/k}]), \quad u(0) = 0 \tag{2.24}$$

and

$$v = kt(1+v)^{(\beta+k)/k}\exp(py^r[1-(1+v)^{r/k}]), \quad v(0) = 0. \tag{2.25}$$

In their special cases when $p = r = 1$, (2.22) and (2.23) obviously yield the following generating functions for the Konhauser polynomials $Y_n^\alpha(x; k)$:

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\beta n}(x + ny; k)t^n = \frac{(1-\xi)^{-(\alpha+1)/k}\exp(x[1-(1-\xi)^{-1/k}])}{1 - k^{-1}\xi(1-\xi)^{-1}[\beta - y(1-\xi)^{-1/k}]}, \tag{2.26}$$

where $\xi$ is a function of $t$ defined *implicitly* by

$$\xi = t(1-\xi)^{-\beta/k}\exp(y[1-(1-\xi)^{-1/k}]), \quad \xi(0) = 0; \tag{2.27}$$

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\beta n}(x + ny; k)t^n = \frac{(1+\eta)^{(\alpha+1)/k}\exp(x[1-(1+\eta)^{1/k}])}{1 - k^{-1}\eta[\beta - y(1+\eta)^{1/k}]}, \tag{2.28}$$

where $\eta$ is a function of $t$ given *implicitly* by

$$\eta = t(1+\eta)^{(\beta+k)/k}\exp(y[1-(1+\eta)^{1/k}]), \quad \eta(0) = 0. \tag{2.29}$$

For $y = 0$, the generating functions (2.26) and (2.28) are essentially equivalent to the Calvez-Génin result [2, p. A41, Eq. (2)]. {Indeed, their reductions to (2.15) when $\beta = y = 0$ and to (2.16) when

$\beta = -k$ and $y = 0$ are immediate.} On the other hand, their special cases when $k = 1$, involving Laguerre polynomials, were given recently by Carlitz [4, p. 525, Eq. (5.2) and (5.5)].

From the Srivastava-Singhal result [24, p. 78, Eq. (3.2)] we further have

$$\text{(2.30)}\quad \begin{aligned} &(x^{1-r}D_x)^m\{\exp(-px^r)G_n^{(\alpha)}(x, r, p, k)\} \\ &\qquad = (-rp)^m \exp(-px^r)G_n^{(\alpha+mr)}(x, r, p, k)\,, \quad m \geqq 0\,, \end{aligned}$$

and

$$\text{(2.31)}\quad G_n^{(\alpha+\beta)}([x^r + y^r]^{1/r}, r, p, k) = \sum_{j=0}^{n} G_j^{(\alpha)}(x, r, p, k)\, G_{n-j}^{(\beta)}(y, r, p, k)\,,$$

which, for $p = r = 1$, yield the known results

$$\text{(2.32)}\quad D_x^m\{e^{-x}Y_n^\alpha(x; k)\} = (-1)^m e^{-x} Y_n^{\alpha+m}(x; k)\,, \quad m \geqq 0$$

and

$$\text{(2.33)}\quad Y_n^{\alpha+\beta+1}(x + y; k) = \sum_{j=0}^{n} Y_j^\alpha(x; k) Y_{n-j}^\beta(y; k)\,,$$

due to Génin *et* Calvez [8, p. A34, Eq. (6); p. A33, Eq. (2)]. {For (2.33) see also [15, p. 803, Eq. (3.2)].}

Applying (2.30) in conjunction with Taylor's theorem, we obtain yet another new generating function in the form:

$$\text{(2.34)}\quad \sum_{n=0}^{\infty} G_m^{(\alpha+nr)}(x, r, p, k)\frac{t^n}{n!} = e^t\, G_m^{(\alpha)}([x^r - t/p]^{1/r}, r, p, k)\,, \quad m \geqq 0\,,$$

which, in view of the relationship (2.3), reduces at once to the Génin-Calvez result [8, p. A34, Eq. (7)]

$$\text{(2.35)}\quad \sum_{n=0}^{\infty} Y_m^{\alpha+n}(x; k)\frac{t^n}{n!} = e^t Y_m^\alpha(x - t; k)\,, \quad m \geqq 0\,.$$

We conclude this part by recording the following special case of a known result given by Srivastava and Singhal [24, p. 84, Eq. (7.3)]:

$$\text{(2.36)}\quad Y_n^\alpha(x; k) = \sum_{j=0}^{n} \binom{j - 1 + (\alpha - \beta)/k}{j} Y_{n-j}^\beta(x; k)\,,$$

which is due to Prabhakar [15, p. 802, Eq. (3.1)]; for $k = 1$, (2.36) yields a well-known property of the Laguerre polynomials [18, p. 209, Eq. (2)].

Incidentally, the well-known special case $y = 0$ of (2.28) [with $\beta$ replaced *trivially* by $kb$], and an *erroneous* version of the Génin-Calvez result (2.35), were rederived in a recent paper by B. K. Karande and K. R. Patil [*Indian J. Pure Appl. Math.* 12 (1981),

222–225; especially see p. 224, Eq. (12), and p. 223, Eq. (6)] *without* any reference to the relevant earlier papers [2], [7] and [8].

V. *Mixed multilateral generating functions.* The generating-function relationships (2.17) and (2.19) enable us to apply the results of Srivastava and Lavoie [23], and we are led rather immediately to the following interesting variations of a general bilateral generating function [*op. cit.*, p. 319, Eq. (107)]:

$$\begin{aligned}\sum_{n=0}^{\infty} Y_{m+n}^{\alpha}(x;k)\Lambda_n(y_1,\cdots,y_N;z)t^n \\ = (1-t)^{-(km+\alpha+1)/k}\exp(x[1-(1-t)^{-1/k}]) \\ \times F[x(1-t)^{-1/k};y_1,\cdots,y_N;zt^q/(1-t)^q]\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} Y_{m+n}^{\alpha-kn}(x;k)\Lambda_n(y_1,\cdots,y_N;z)t^n \\ = (1+t)^{(\alpha-k+1)/k}\exp(x[1-(1+t)^{1/k}]) \\ \times G[x(1+t)^{1/k};y_1,\cdots,y_N;zt^q/(1+t)^q] ,\end{aligned} \tag{2.38}$$

where

$$F[x;y_1,\cdots,y_N;z] = \sum_{n=0}^{\infty} c_n Y_{m+qn}^{\alpha}(x;k)\Delta_n(y_1,\cdots,y_N)z^n , \tag{2.39}$$

$$G[x;y_1,\cdots,y_N;z] = \sum_{n=0}^{\infty} c_n Y_{m+qn}^{\alpha-kqn}(x;k)\Delta_n(y_1,\cdots,y_N)z^n , \tag{2.40}$$

$c_n \neq 0$ are arbitrary complex constants, $m \geqq 0$ and $q \geqq 1$ are integers, and, in terms of the non-vanishing functions $\Delta_n(y_1,\cdots,y_N)$ of $N$ variables $y_1,\cdots,y_N$, $N \geqq 1$,

$$\Lambda_n(y_1,\cdots,y_N;z) = \sum_{j=0}^{[n/q]} \binom{m+n}{n-qj} c_j \Delta_j(y_1,\cdots,y_N)z^j . \tag{2.41}$$

By assigning suitable values to the arbitrary coefficients $c_n$, it is fairly straightforward to derive, from the general formulas (2.37) and (2.38), a considerably large variety of bilateral generating functions for the polynomials $Y_n^{\alpha}(x;k)$ and $Y_n^{\alpha-kn}(x;k)$, respectively. On the other hand, in every situation in which the multivariable function $\Delta_n(y_1,\cdots,y_N)$ can be expressed as a suitable product of several simpler functions, we shall be led to an interesting class of mixed multilateral generating functions for the Konhauser polynomials considered and, of course, for the Laguerre polynomials when $k=1$, and for the polynomial systems studied by Spencer and Fano [19] and Preiser [16] when $k=2$.

VI. *Further finite sums.* The results to be presented here are in addition to the finite summation formulas (2.33) and (2.36) and their general forms involving the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, r, p, k)$. Indeed, from the known generating functions [24, p. 78, Eq. (3.2); p. 79, Eq. (3.4)] it is readily observed that

$$G_n^{(\alpha)}(x, r, p, k) = \sum_{j=0}^{n-1} (-k)^j \binom{n-1}{j} G_{n-j}^{(\alpha-k+kn)}(x, r, p, k) , \tag{2.42}$$

$$G_n^{(\alpha)}(x, r, p, k) = \sum_{j=0}^{n-1} k^j \binom{n-1}{j} G_{n-j}^{(\alpha+k-kn+kj)}(x, r, p, k) \tag{2.43}$$

and

$$G_n^{(\alpha)}(x, r, p, k) = \sum_{j=0}^{n} k^j \binom{(\alpha-\beta)/k}{j} G_{n-j}^{(\beta+kj)}(x, r, p, k) , \tag{2.44}$$

which, on setting $p = r = 1$ and appealing to (2.3), yield the following new results involving the Konhauser polynomials $Y_n^\alpha(x; k)$:

$$Y_n^\alpha(x; k) = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} Y_{n-j}^{\alpha-k+kn}(x; k) , \tag{2.45}$$

$$Y_n^\alpha(x; k) = \sum_{j=0}^{n-1} \binom{n-1}{j} Y_{n-j}^{\alpha+k-kn+kj}(x; k) \tag{2.46}$$

and

$$Y_n^\alpha(x; k) = \sum_{j=0}^{n} \binom{(\alpha-\beta)/k}{j} Y_{n-j}^{\beta+kj}(x; k) , \tag{2.47}$$

respectively.

This last formula (2.47) is analogous to the earlier result (2.36).

3. **The biorthogonal polynomials $Z_n^\alpha(x; k)$.** Since the parameter $k$ in (1.2) is restricted, by definition, to take on positive integer values, by the well-known multiplication theorem for the $\Gamma$-function we have

$$\Gamma(kj + \alpha + 1) = \Gamma(\alpha + 1) \prod_{i=1}^{k} \left(\frac{\alpha + i}{k}\right)_j , \quad j = 0, 1, 2, \cdots , \tag{3.1}$$

where $(\lambda)_n$ is given by (1.4). From (1.2) and (3.1) we obtain the hypergeometric representation

$$Z_n^\alpha(x; k) = \frac{(\alpha+1)_{kn}}{n!} \, {}_1F_k[-n; (\alpha+1)/k, \cdots, (\alpha+k)/k; (x/k)^k] , \tag{3.2}$$

which can alternatively be used to derive the following properties

of the biorthogonal polynomials $Z_n^\alpha(x; k)$ by simply specializing those of the generalized hypergeometric function

$$ {}_pF_q[\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_m}{\prod_{j=1}^{q} (\beta_j)_m} \frac{z^m}{m!}, \tag{3.3}$$

where $\beta_j \neq 0, -1, -2, \cdots, \quad \forall j \in \{1, \cdots, q\}$.

I. *Differential equations.* Denoting the first member of the preceding equation (3.3) by $F$, we have the well-known hypergeometric differential equation [18, p. 77, Eq. (2)]

$$\left[\theta \prod_{j=1}^{q} (\theta + \beta_j - 1) - z \prod_{j=1}^{p} (\theta + \alpha_j)\right] F = 0 , \quad p \leqq q + 1 , \tag{3.4}$$

where, for convenience, $\theta = zD_z$.

In (3.4) we set $p = 1$, $q = k$, $z = (x/k)^k$, $\theta = k^{-1}\delta$, where $\delta = xD_x$, and apply the hypergeometric representation (3.2). We thus obtain a differential equation satisfied by the polynomials $Z_n^\alpha(x; k)$ in the form:

$$\left\{\prod_{j=1}^{k} (\delta + \alpha - k + j)\right\} \delta Z_n^\alpha(x; k) = x^k(\delta - kn) Z_n^\alpha(x; k) . \tag{3.5}$$

Recalling that (*cf.*, *e.g.*, [26, p. 310, Eq. (19)])

$$f(\delta + \alpha)\{g(x)\} = x^{-\alpha} f(\delta)\{x^\alpha g(x)\} , \quad \delta = xD_x , \tag{3.6}$$

it is easily verified that

$$\prod_{j=1}^{k} (\delta + \alpha - k + j)\{g(x)\} = x^{k-\alpha} D_x^k\{x^\alpha g(x)\} , \tag{3.7}$$

and the differential equation (3.5) obviously reduces to its equivalent form [10, p. 306, Eq. (10)]

$$D_x^k\{x^{\alpha+1} D_x Z_n^\alpha(x; k)\} = x^\alpha(xD_x - kn) Z_n^\alpha(x; k) . \tag{3.8}$$

II. *Recurrence relations.* It is well known that (*cf.*, *e.g.*, [11, p. 279, Problem 20])

$$\begin{aligned} &D_z\{{}_pF_q[\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z]\} \\ &= \frac{\alpha_1 \cdots \alpha_p}{\beta_1 \cdots \beta_q} {}_pF_q[\alpha_1 + 1, \cdots, \alpha_p + 1; \beta_1 + 1, \cdots, \beta_q + 1; z] , \end{aligned} \tag{3.9}$$

whence, by setting $p = 1$, $q = k$, $z = (x/k)^k$, $D_z = (k/x)^{k-1} D_x$, and applying (3.2), we have

$$D_x Z_n^\alpha(x; k) = -kx^{k-1} Z_{n-1}^{\alpha+k}(x; k) , \tag{3.10}$$

or, more generally,

(3.11) $$(x^{1-k}D_x)^m Z_n^\alpha(x;k) = (-k)^m Z_{n-m}^{\alpha+km}(x;k), \quad n \geqq m \geqq 0.$$

Similarly, from the known results ([18, p. 82, Eq. (12), (13) and (15)]; see also [17]), involving the generalized hypergeometric function (3.3), we readily obtain the following mixed recurrence relations:

(3.12) $$xD_x Z_n^\alpha(x;k) = knZ_n^\alpha(x;k) - \frac{k\Gamma(kn+\alpha+1)}{\Gamma(k(n-1)+\alpha+1)} Z_{n-1}^\alpha(x;k),$$

(3.13) $$xD_x Z_n^\alpha(x;k) = (kn+\alpha)Z_n^{\alpha-1}(x;k) - \alpha Z_n^\alpha(x;k),$$

(3.14) $$Z_n^\alpha(x;k) - Z_n^{\alpha-1}(x;k) = \frac{k\Gamma(kn+\alpha)}{\Gamma(k(n-1)+\alpha+1)} Z_{n-1}^\alpha(x;k).$$

It is not difficult to verify that the recurrence relation (3.14) results from (3.12) and (3.13) by eliminating their common term $xD_x Z_n^\alpha(x;k)$. If, however, we eliminate this derivative term in (3.12) or (3.13) by using (3.10) instead, we shall arrive at the recurrence relations

(3.15) $$x^k Z_n^{\alpha+k}(x;k) = (kn+\alpha+1)_k Z_n^\alpha(x;k) - (n+1)Z_{n+1}^\alpha(x;k)$$

and[1]

(3.16) $$kx^k Z_n^{\alpha+k}(x;k) = \alpha Z_{n+1}^\alpha(x;k) - (kn+\alpha+k)Z_{n+1}^{\alpha-1}(x;k).$$

Formulas[2] (3.10) and (3.12) were given earlier by Konhauser [10, p. 306, Eq. (8); p. 305, Eq. (6)], (3.14) is due to Génin *et* Calvez [7, p. A1565, Eq. (5)], while (3.15) was derived by Prabhakar [14, p. 215, Eq. (2.6)] by using a contour integral representation for $Z_n^\alpha(x;k)$.

For a direct proof of (3.15), we observe from (1.2) that

$$\begin{aligned} x^k Z_n^{\alpha+k}(x;k) &= \frac{\Gamma(k(n+1)+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{k(j+1)}}{\Gamma(k(j+1)+\alpha+1)} \\ &= \frac{\Gamma(k(n+1)+\alpha+1)}{n!} \sum_{j=1}^{n+1} (-1)^{j-1} \binom{n}{j-1} \frac{x^{kj}}{\Gamma(kj+\alpha+1)}, \end{aligned}$$

and since

$$-\binom{n}{j-1} = \binom{n}{j} - \binom{n+1}{j}, \quad 0 \leqq j \leqq n+1,$$

it follows that

[1] The pure recurrence relation (3.16) appears *erroneously* in a recent paper by K. R. Patil and N. K. Thakare [*J. Mathematical Phys.* **18** (1977), 1724-1726; especially see p. 1725].

[2] It may be of interest to mention here that the known results (3.10) and (3.15) were rederived, using Prabhakar's version [14, p. 214, Eq. (2.2)] of the generating function (3.20) of this paper, by B. Nath [*Kyungpook Math. J.* **14** (1974), 81-82].

$$x^k Z_n^{\alpha+k}(x;k)$$
$$= \frac{\Gamma(k(n+1)+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)}$$
$$- \frac{\Gamma(k(n+1)+\alpha+1)}{n!}\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)}$$
$$= (kn+\alpha+1)_k Z_n^\alpha(x;k) - (n+1)Z_{n+1}^\alpha(x;k)\,,$$

ich precisely is the pure recurrence relation (3.15).

III. *Generating functions.* Chaundy [5] has shown that [*op.* ., p. 62, Eq. (25)]

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{p+1}F_q[-n,\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z]t^n$$
17)
$$= (1-t)^{-\lambda}\,{}_{p+1}F_q[\lambda,\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;zt/(t-1)]\,,\quad |t|<1\,.$$

If we replace $t$ on both sides of (3.17) by $t/\lambda$ and take their nits as $\lambda\to\infty$, we shall readily obtain Rainville's result:

$$\sum_{n=0}^{\infty}{}_{p+1}F_q[-n,\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z]\frac{t^n}{n!}$$
.18)
$$= e^t\,{}_pF_q[\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;-zt]\,.$$

Both (3.17) and (3.18) are stated by Erdélyi *et al.* [6, p. 267, q. (22) and (25)], and their various generalizations have appeared ı the literature (*cf.*, *e.g.*, [20, p. 68, Eq. (3.9) and (3.10)].

By specializing (3.17) and (3.18) in view of the hypergeometric epresentation (3.2) for $Z_n^\alpha(x;k)$, we at once get the generating unctions

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(\alpha+1)_{kn}}Z_n^\alpha(x;k)t^n$$
3.19)
$$= (1-t)^{-\lambda}\,{}_1F_k\left[\lambda;\frac{\alpha+1}{k},\cdots,\frac{\alpha+k}{k};\frac{x^k t}{(t-1)k^k}\right],\quad |t|<1$$

and

$$\sum_{n=0}^{\infty}Z_n^\alpha(x;k)\frac{t^n}{(\alpha+1)_{kn}}$$
(3.20)
$$= e^t\,{}_0F_k\left[-;\frac{\alpha+1}{k},\cdots,\frac{\alpha+k}{k};-\left(\frac{x}{k}\right)^k t\right],$$

respectively.

The generating function (3.19) is due essentially to Génin *et* Calvez [7, p. A1564, Eq. (3)], while (3.20) was given by Srivastava

[21, p. 490, Eq. (7)]; the latter appears also, with an obvious typographical error, in a recent paper [12, p. 922]. In fact, both (3.19) and (3.20) were given (in disguised forms) by Prabhakar [14, p. 218, Eq. (4.1); p. 214, Eq. (2.2)]. Notice that the so-called generalized Mittag-Leffler function $E^{\lambda}_{k,\alpha+1}(z)$ and the "Bessel-Maitland" function[3] $\phi(k, \alpha+1; z)$, occurring in Prabhakar's results just cited, are indeed the familiar hypergeometric functions ${}_1F_k$ and ${}_0F_k$, respectively, $k$ being a positive integer. More precisely, we have, for $k = 1, 2, 3, \cdots$,

$$\begin{aligned} E^{\lambda}_{k,\alpha+1}(z) &= \sum_{m=0}^{\infty} \frac{(\lambda)_m z^m}{m!\,\Gamma(km+\alpha+1)} \\ &= \frac{1}{\Gamma(\alpha+1)}\,{}_1F_k\left[\lambda;\ \frac{\alpha+1}{k}, \cdots, \frac{\alpha+k}{k};\ \left(\frac{z}{k}\right)^k\right] \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \phi(k, \alpha+1; z) &= \sum_{m=0}^{\infty} \frac{z^m}{m!\,\Gamma(km+\alpha+1)} \\ &= \frac{1}{\Gamma(\alpha+1)}\,{}_0F_k\left[-;\ \frac{\alpha+1}{k}, \cdots, \frac{\alpha+k}{k};\ \left(\frac{z}{k}\right)^k\right], \end{aligned} \tag{3.22}$$

by appealing to the well-known multiplication theorem for the $\Gamma$-function.

Next we consider the double series

$$\begin{aligned} &\sum_{m=0}^{\infty} z^m \sum_{n=0}^{\infty} \binom{m+n}{n} Z^{\alpha}_{m+n}(x;k) \frac{t^n}{(\alpha+1)_{k(m+n)}} \\ &\quad = \sum_{n=0}^{\infty} \frac{Z^{\alpha}_n(x;k)}{(\alpha+1)_{kn}} \sum_{m=0}^{n} \binom{n}{m} t^{n-m} z^m = \sum_{n=0}^{\infty} Z^{\alpha}_n(x;k) \frac{(z+t)^n}{(\alpha+1)_{kn}} \\ &\quad = e^{z+t}\,{}_0F_k\left[-;\ \frac{\alpha+1}{k}, \cdots, \frac{\alpha+k}{k};\ -\left(\frac{x}{k}\right)^k (z+t)\right], \quad \text{by (3.20)}, \\ &\quad = \sum_{n,\nu=0}^{\infty} \frac{(-x^k)^n}{n!\,\nu!\,(\alpha+1)_{kn}} \sum_{m=0}^{n+\nu} \binom{n+\nu}{m} t^{n+\nu-m} z^m \\ &\quad = \sum_{m=0}^{\infty} z^m \sum_{n+\nu \geqq m} \binom{n+\nu}{m} \frac{t^{n-m}}{n!} \frac{(-x^k)^n}{(\alpha+1)_{kn}} \frac{t^{\nu}}{\nu!}, \end{aligned}$$

and, on equating the coefficients of $z^m$, we have the generating relation

$$\begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n}{n} Z^{\alpha}_{m+n}(x;k) \frac{t^n}{(\alpha+1)_{k(m+n)}} \\ &\quad = \sum_{n=m}^{\infty} \binom{n}{m} \frac{t^{n-m}}{n!} \frac{(-x^k)^n}{(\alpha+1)_{kn}}\,{}_1F_1[n+1;\ n-m+1;\ t], \end{aligned} \tag{3.23}$$

[3] Incidentally, the generalized Bessel function $\phi(\alpha, \beta; z)$ was introduced by E. Maitland Wright [27, p. 72, Eq. (1.3)]; see also Erdélyi *et al.* [6, p. 211, Eq. (27)].

which holds true for every non-negative integer $m$.

Alternatively, this last generating relation (3.23) may be derived as a special case of our earlier result [20, p. 68, Theorem 3]. Of course, it is not difficult to develop a *direct* proof of (3.23) without using the generating function (3.20).

For $m = 0$, (3.23) evidently reduces to the familiar generating function (3.20). Its special case when $k = 1$ leads to what is obviously contained in the following limiting form of a known result [22, p. 152, Eq. (19)]:

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha)}(x) \frac{t^n}{(\mu)_n} \\ = \binom{\alpha+m}{m} e^x \Psi_2[\alpha+m+1; \mu, \alpha+1; t, -x] ,\end{aligned} \tag{3.24}$$

where $\Psi_2$ is a (Humbert's) confluent hypergeometric function of two variables defined by [1, p. 126]

$$\Psi_2[a; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!} . \tag{3.25}$$

Formula (3.24) follows from the known generating function [22, p. 152, Eq. (19)] by writing $t/\lambda$ in place of $t$ and then letting $\lambda \to \infty$. Furthermore, if we replace $t$ in (3.24) by $\mu t$ and let $\mu \to \infty$, we shall arrive at the well-known generating function [18, p. 211, Eq. (9)]

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha)}(x) t^n = (1-t)^{-m-\alpha-1} \exp\left(-\frac{xt}{1-t}\right) \\ \times L_m^{(\alpha)}\left(\frac{x}{1-t}\right), \quad m = 0, 1, 2, \cdots ,\end{aligned} \tag{3.26}$$

which follows also from (2.17) when $k = 1$.

IV. *Multilinear generating functions.* By making use of the hypergeometric representation (3.2), a number of new multilinear generating functions for the product

$$Z_{n_1}^{\alpha_1}(y_1; k_1) \cdots Z_{n_r}^{\alpha_r}(y_r; k_r) , \tag{3.27}$$

analogous to the *corrected* version of the Patil-Thakare result [12, p. 921, Eq. (2.1)], can be derived by suitably specializing a general formula given earlier by Srivastava and Singhal [25, p. 1244, Eq. (24)] for a product of several generalized hypergeometric polynomials. We omit the details involved.

V. *Finite summation formulas.* In view of the exponential

generating function (3.20), Theorem 1 (p. 64) of Srivastava [20] will apply to the biorthogonal polynomials $Z_n^\alpha(x;k)$, and we thus have

$$(3.28)\qquad Z_n^\alpha(x;k) = \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^{n} \binom{\alpha+kn}{kj} \frac{(kj)!}{j!} \left(\frac{y^k - x^k}{x^k}\right)^j Z_{n-j}^\alpha(y;k)\,,$$

or, equivalently,

$$(3.29)\qquad Z_n^\alpha(x;k) = \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^{n} \binom{\alpha+kn}{kn-kj} \frac{(kn-kj)!}{(n-j)!} \left(\frac{y^k - x^k}{x^k}\right)^{n-j} Z_j^\alpha(y;k)\,.$$

The summation formula (3.28) can indeed be derived directly (*cf.* [21, p. 490, § 4]). It can also be rewritten in the form [*op. cit.*, p. 491, Eq. (12)]:

$$(3.30)\qquad Z_n^\alpha(\mu x;k) = \sum_{j=0}^{n} \binom{kn+\alpha}{kj} \frac{(kj)!}{j!} \mu^{k(n-j)} (1-\mu^k)^j Z_{n-j}^\alpha(x;k)\,,$$

which obviously provides us with an elegant multiplication formula for the biorthogonal polynomials $Z_n^\alpha(x;k)$.

VI. *Laplace transforms.* Employing the usual notation for Laplace's transform, *viz*

$$(3.31)\qquad \mathscr{L}\{f(t): s\} = \int_0^\infty e^{-st} f(t)\,dt\,, \qquad \operatorname{Re}(s-\sigma) > 0\,,$$

where $f \in L(0, R)$ for every $R > 0$, and $f(t) = O(e^{\sigma t})$, $t \to \infty$, we have

$$(3.32)\qquad \begin{aligned} &\mathscr{L}\{t^\beta Z_n^\alpha(xt;k): s\} \\ &= \frac{(\alpha+1)_{kn}\,\Gamma(\beta+1)}{s^{\beta+1}\, n!} \\ &\quad \times {}_{k+1}F_k\left[-n, \frac{\beta+1}{k}, \ldots, \frac{\beta+k}{k}; \frac{\alpha+1}{k}, \ldots, \frac{\alpha+k}{k}; \left(\frac{x}{s}\right)^k\right], \end{aligned}$$

provided that $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(\beta) > -1$.

The Laplace transform formula (3.32) can be derived fairly easily from the hypergeometric representation (3.2) by using readily available tables. In the special case when $\beta = \alpha$, it simplifies at once to the elegant form [14, p. 217, Eq. (3.7)]:

$$(3.33)\qquad \mathscr{L}\{t^\alpha Z_n^\alpha(xt;k): s\} = \frac{\Gamma(kn+\alpha+1)}{s^{kn+\alpha+1}\, n!}(s^k - x^k)^n\,,$$

where, as before, $\operatorname{Re}(s) > 0$ and (by definition) $\alpha > -1$.

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Received September 3, 1980. This work was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant A-7353.

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# Pacific Journal of Mathematics

PACIFIC JOURNAL OF MATHEMATICS
Vol. 117, No. 1, 1985

# A MULTILINEAR GENERATING FUNCTION FOR THE KONHAUSER SETS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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The polynomial sets $\{Y_n^\alpha(x;k)\}$ and $\{Z_n^\alpha(x;k)\}$, discussed by Joseph D. E. Konhauser, are biorthogonal over the interval $(0,\infty)$ with respect to the weight function $x^\alpha e^{-x}$, where $\alpha > -1$ and $k$ is a positive integer. The object of the present note is to develop a fairly elementary method of proving a general multilinear generating function which, upon suitable specializations, yields a number of interesting results including, for example, a multivariable hypergeometric generating function for the multiple sum:

$$(*)\qquad \sum_{n_1,\ldots,n_r=0}^{\infty} (m+n_1+\cdots+n_r)!\,Y^{\alpha}_{m+n_1+\cdots+n_r}(x;k) \cdot \prod_{i=1}^{r}\left\{\frac{Z^{\beta_i}_{n_i}(y_i;s_i)\,u_i^{n_i}}{(1+\beta_i)_{s_i n_i}}\right\},$$

involving the Konhauser biorthogonal polynomials; here, by definition,

$$\alpha > -1;\quad \beta_i > -1;\quad k, s_i = 1,2,3,\ldots;\qquad \forall\, i \in \{1,\ldots,r\}.$$

**1. Introduction.** Joseph D. E. Konhauser ([5]; see also [4]) introduced two interesting classes of polynomials: $Y_n^\alpha(x;k)$ a polynomial in $x$, and $Z_n^\alpha(x;k)$ a polynomial in $x^k$, $\alpha > -1$ and $k = 1,2,3,\ldots$. For $k=1$, these polynomials reduce to the classical Laguerre polynomials $L_n^{(\alpha)}(x)$, and for $k=2$ they were encountered earlier by Spencer and Fano [8] in the study of the penetration of gamma rays through matter and were discussed subsequently by Preiser [7]. Also [5, p. 303]

$$(1)\qquad \int_0^\infty x^\alpha e^{-x}\, Y_m^\alpha(x;k)\,Z_n^\alpha(x;k)\,dx = \frac{\Gamma(kn+\alpha+1)}{n!}\delta_{mn},\qquad \forall\, m,n \in \{0,1,2,\ldots\},$$

so that the Konhauser polynomial sets $\{Y_n^\alpha(x;\, k)\}$ and $\{Z_n^\alpha(x;\, k)\}$ are *biorthogonal* over the interval $(0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$, where $\alpha > -1$, $k$ is a positive integer, and $\delta_{mn}$ is the Kronecker delta.

The following explicit expression for the polynomials $Z_n^\alpha(x;\, k)$ was given by Konhauser [5, p. 304, Eq. (5)]:

$$(2) \qquad Z_n^\alpha(x;\, k) = \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)}.$$

Subsequently, Carlitz pointed out that [2, p. 427, Eq. (9)]

$$(3) \qquad Y_n^\alpha(x;\, k) = \frac{1}{n!} \sum_{j=0}^{n} \frac{x^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{l+\alpha+1}{k}\right)_n,$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$.

In a recent paper [10] we derived various properties of (for example) the Konhauser biorthogonal polynomials $Y_n^\alpha(x;\, k)$ by suitably specializing those of the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, h, p, k)$ which are defined by the generalized Rodrigues formula [14, p. 75, Eq. (1.3)]

$$(4) \qquad G_n^{(\alpha)}(x, h, p, k) = \frac{x^{-kn-\alpha}\exp(px^h)}{n!} \cdot (x^{k+1}D_x)^n \{x^\alpha \exp(-px^h)\}, \qquad D_x = \frac{d}{dx},$$

and given explicitly by [14, p. 77, Eq. (2.1)]

$$(5) \qquad G_n^{(\alpha)}(x, h, p, k) = \frac{k^n}{n!} \sum_{j=0}^{n} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_n,$$

where the parameters $\alpha$, $h$, $k$ and $p$ are unrestricted, in general. In fact, by comparing (5) with Carlitz's result (3), we at once deduce the known relationship [13, p. 315, Eq. (83)]

$$(6) \qquad Y_n^\alpha(x;\, k) = k^{-n} G_n^{(\alpha+1)}(x, 1, 1, k), \qquad \alpha > -1;\ k = 1, 2, 3, \ldots,$$

which was of fundamental importance in our paper [10].

The object of the present note is first to give a rather elementary proof of a general multilinear generating function for the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, h, p, k)$. We then show how this multilinear generating function can be further generalized and applied to derive a

number of interesting results including, for example, a multivariable hypergeometric generating function for the multiple sum (*) involving the product of several Konhauser biorthogonal polynomials. Our main result is contained in the following

THEOREM. *For a bounded multiple sequence* $\{\Lambda(n_1,\ldots,n_r)\}$ *of arbitrary complex numbers, let*

$$\mathscr{H}[n_1,\ldots,n_r;y_1,\ldots,y_r] = \sum_{j_1=0}^{[n_1/m_1]}\cdots\sum_{j_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1j_1}}{j_1!}\cdots\frac{(-n_r)_{m_rj_r}}{j_r!}\cdot\Lambda(j_1,\ldots,j_r)y_1^{j_1}\cdots y_r^{j_r}, \tag{7}$$

*where* $m_1,\ldots,m_r$ *are positive integers. Also let* $\Delta_r$ *be defined by*

$$\Delta_r = 1-\sum_{i=1}^{r}u_i,\qquad r=1,2,3,\ldots. \tag{8}$$

*Then, for every nonnegative integer* $m$,

$$\begin{aligned}&\sum_{n_1,\ldots,n_r=0}^{\infty}(m+n_1+\cdots+n_r)!G^{(\alpha)}_{m+n_1+\cdots+n_r}(x,h,p,k)\\&\cdot\mathscr{H}[n_1,\ldots,n_r;y_1,\ldots,y_r]\frac{(u_1/k)^{n_1}}{n_1!}\cdots\frac{(u_r/k)^{n_r}}{n_r!}\\&=k^m\exp(px^h)\Delta_r^{-m-\alpha/k}\\&\cdot\sum_{n,n_1,\ldots,n_r=0}^{\infty}\left(\frac{hn+\alpha}{k}\right)_{m+m_1n_1+\cdots+m_rn_r}\left(\frac{1}{n!}\right)\Lambda(n_1,\ldots,n_r)\left(-\frac{px^h}{\Delta_r^{h/k}}\right)^n\\&\cdot\prod_{i=1}^{r}\left\{\frac{[(-u_i/\Delta_r)^{m_i}y_i]^{n_i}}{n_i!}\right\},\qquad k\neq 0,\end{aligned} \tag{9}$$

*provided that the multiple series on the right-hand side of* (9) *has a meaning, and*

$$|u_1+\cdots+u_r|<1. \tag{10}$$

**2. Proof of the theorem.** For convenience, let $\Omega(u_1,\ldots,u_r)$ denote the left-hand side of (9), and set

$$N=n_1+\cdots+n_r\quad\text{and}\quad J=m_1j_1+\cdots+m_rj_r. \tag{11}$$

Applying the explicit representation (5) and the definition (7), we find that

$$(12)\qquad \Omega(u_1,\dots,u_r) = k^m \sum_{n_1,\dots,n_r=0}^{\infty} u_1^{n_1}\cdots u_r^{n_r}$$

$$\cdot \sum_{j=0}^{m+N} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+N}$$

$$\cdot \prod_{i=1}^{r} \left\{ \sum_{j_i=0}^{[n_i/m_i]} \frac{[(-1)^{m_i} y_i]^{j_i}}{j_i!(n_i - m_i j_i)!} \right\} \Lambda(j_1,\dots,j_r)$$

$$= k^m \sum_{j_1,\dots,j_r=0}^{\infty} \Lambda(j_1,\dots,j_r) \prod_{i=1}^{r} \left\{ \frac{[(-u_i)^{m_i} y_i]^{j_i}}{j_i!} \right\}$$

$$\cdot \sum_{n_1,\dots,n_r=0}^{\infty} \frac{u_1^{n_1}}{n_1!} \cdots \frac{u_r^{n_r}}{n_r!} \sum_{j=0}^{m+N+J} \frac{(px^h)^j}{j!}$$

$$\cdot \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+N+J}.$$

Now we appeal to the series identity [9, p. 4, Eq. (12)]

$$(13)\qquad \sum_{n_1,\dots,n_r=0}^{\infty} f(n_1 + \cdots + n_r) \frac{u_1^{n_1}}{n_1!} \cdots \frac{u_r^{n_r}}{n_r!}$$

$$= \sum_{n=0}^{\infty} f(n) \frac{(u_1 + \cdots + u_r)^n}{n!},$$

and (12) becomes

$$(14)\qquad \Omega(u_1,\dots,u_r) = k^m \sum_{n,j_1,\dots,j_r=0}^{\infty} \frac{(u_1 + \cdots + u_r)^n}{n!}$$

$$\cdot \prod_{i=1}^{r} \left\{ \frac{[(-u_i)^{m_i} y_i]^{j_i}}{j_i!} \right\} \sum_{j=0}^{m+n+J} \frac{(px^h)^j}{j!}$$

$$\cdot \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{hl+\alpha}{k}\right)_{m+n+J},$$

where $J$ is defi: ed, as before, by (11).

The innermost sum in (14) is the $j$th difference of a polynomial of degree $m + n + J$ in $\alpha$; it is nil when $j > m + n + J$. Thus we have

$$\sum_{j=0}^{m+n+J} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \binom{\frac{hl+\alpha}{k}}{m+n+J}$$

$$= \sum_{l=0}^{\infty} \binom{\frac{hl+\alpha}{k}}{m+n+J} \frac{(-px^h)^l}{l!} \sum_{j=0}^{\infty} \frac{(px^h)^j}{j!}$$

$$= \exp(px^h) \sum_{l=0}^{\infty} \binom{\frac{hl+\alpha}{k}}{m+n+J} \frac{(-px^h)^l}{l!},$$

and substituting this expression in (14), and applying the binomial expansion to sum the resulting $n$-series, we finally obtain

$$(15)\quad \Omega(u_1,\ldots,u_r) = k^m \exp(px^h) \Delta_r^{-m-\alpha/k}$$

$$\cdot \sum_{l,j_1,\ldots,j_r=0}^{\infty} \binom{\frac{hl+\alpha}{k}}{m+J} \left(\frac{1}{l!}\right) \Delta(j_1,\ldots,j_r) \left(-\frac{px^h}{\Delta_r^{h/k}}\right)^l$$

$$\cdot \prod_{i=1}^{r} \left\{ \frac{[(-u_i/\Delta_r)^{m_i} y_i]^{j_i}}{j_i!} \right\}, \qquad k \neq 0,$$

where $\Delta_r$ and $J$ are given by (8) and (11), respectively, and the inequality in (10) is assumed to hold.

The right-hand sides of (9) and (15) are essentially the same. This evidently completes the proof of our theorem under the hypothesis that the various interchanges of the order of summation are permissible by absolute convergence of the series involved. Thus, in general, our theorem holds true whenever each member of (9) has a meaning.

REMARK. Our method of derivation can be applied *mutatis mutandis* in order to prove the following generalization of the multilinear generating function (9):

$$(16)\quad \sum_{n_1,\ldots,n_r=0}^{\infty} (m+n_1+\cdots+n_r)!\,\mathscr{F}_{m+n_1+\cdots+n_r}^{(\alpha)}(x,h,p,k)$$

$$\cdot \mathscr{H}[n_1,\ldots,n_r; y_1,\ldots,y_r] \frac{(u_1/k)^{n_1}}{n_1!} \cdots \frac{(u_r/k)^{n_r}}{n_r!}$$

$$= k^m \exp(px^h) \Delta_r^{-m-\alpha/k} \sum_{n,n_1,\ldots,n_r=0}^{\infty} \binom{\frac{hn+\alpha}{k}}{m+m_1n_1+\cdots+m_rn_r}$$

$$\cdot \frac{\xi_n}{n!} \Delta(n_1,\ldots,n_r) \left(-\frac{px^h}{\Delta_r^{h/k}}\right)^n \prod_{i=1}^{r} \left\{ \frac{[(-u_i/\Delta_r)^{m_i} y_i]^{n_i}}{n_i!} \right\}, \qquad k \neq 0,$$

where, in terms of the bounded sequence $\{\xi_n\}$ of arbitrary complex numbers,

$$(17)\quad \mathscr{F}_n^{(\alpha)}(x, h, p, k) = \frac{k^n}{n!} \sum_{j=0}^{\infty} \frac{(px^h)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \xi_l \left( \frac{hl+\alpha}{k} \right)_n,$$

which obviously reduces to the Srivastava-Singhal equation (5) when $\xi_l = 1, l \geq 0$.

**3. Applications.** By assigning suitable special values to the arbitrary coefficients $\Lambda(j_1, \ldots, j_r)$, the multiple sum in (7) can indeed be expressed in terms of the generalized Lauricella hypergeometric function of $r$ variables [**11**, p. 454]. Thus, following the various notations and conventions explained fairly fully by Srivastava and Daoust ([**11**, p. 545 *et seq.*]; see also [**12**]), we obtain from our theorem the multivariable hypergeometric generating function:

$$(18)\quad \sum_{n_1, \ldots, n_r = 0}^{\infty} (m + n_1 + \cdots + n_r)!\, G^{(\alpha)}_{m+n_1+\cdots+n_r}(x, h, p, k)$$

$$\cdot F^{A:1+B'; \cdots; 1+B^{(r)}}_{C:\; D'; \cdots; D^{(r)}} \left( \begin{matrix} [(a): \theta', \ldots, \theta^{(r)}] : [-n_1 : m_1], & [(b'): \phi']; \cdots; \\ [(c): \psi', \ldots, \psi^{(r)}] : & [(d'): \delta']; \cdots; \end{matrix} \right.$$

$$\left. \begin{matrix} [-n_r : m_r], [(b^{(r)}): \phi^{(r)}]; \\ [(d^{(r)}): \delta^{(r)}]; \end{matrix} \; y_1, \ldots, y_r \right) \left( \frac{u_1}{k} \right)^{n_1} \cdots \left( \frac{u_r}{k} \right)^{n_r}$$

$$= k^m \left( \frac{\alpha}{k} \right)_m \exp(px^h)\, \Delta_r^{-m-\alpha/k}\, F^{1+A:0; B'; \cdots; B^{(r)}}_{C:1; D'; \cdots; D^{(r)}}$$

$$\left( \begin{matrix} [m + \alpha/k : h/k, m_1, \ldots, m_r], [(a): 0, \theta', \ldots, \theta^{(r)}] : \text{———}; \\ [(c): 0, \psi', \ldots, \psi^{(r)}] : [\alpha/k : h/k]; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'): \phi']; \cdots; [(b^{(r)}): \phi^{(r)}]; \\ [(d'): \delta']; \cdots; [(d^{(r)}): \delta^{(r)}]; \end{matrix} \; \Xi_0, \Xi_1, \ldots, \Xi_r \right); \quad k \neq 0,$$

where $h/k > 0$, $\Delta_r$ is given by (8), and

$$(19)\quad \Xi_0 = -\frac{px^h}{\Delta_r^{h/k}}, \qquad \Xi_i = y_i \left( -\frac{u_i}{\Delta_r} \right)^{m_i}, \quad i = 1, \ldots, r.$$

Next we set $A = C = 0$ in (18) and, for convenience, let each of the positive coefficients $\phi_j^{(i)}$, $j = 1, \ldots, B^{(i)}$; $\delta_j^{(i)}$, $j = 1, \ldots, D^{(i)}$ $(i = 1, \ldots, r)$ equal 1. Denoting the array of parameters

$$(-n_i + j - 1)/m_i, \qquad j = 1, \ldots, m_i,$$

by $\Delta(m_i; -n_i)$, $i = 1,\ldots,r$, we thus find from (18) that

$$\text{(20)} \quad \sum_{n_1,\ldots,n_r=0}^{\infty} (m+n_1+\cdots+n_r)!\, G_{m+n_1+\cdots+n_r}^{(\alpha)}(x,h,p,k)$$

$$\cdot \prod_{i=1}^{r} \left\{ {}_{m_i+B^{(i)}}F_{D^{(i)}} \left[ \begin{matrix} \Delta(m_i; -n_i), & (b^{(i)}); \\ & (d^{(i)}); \end{matrix} \; y_i m_i^{m_i} \right] \left( \frac{u_i}{k} \right)^{n_i} \right\}$$

$$= k^m \left( \frac{\alpha}{k} \right)_m \exp(px^h) \Delta_r^{-m-\alpha/k}$$

$$\cdot F^{1:0;B';\cdots;B^{(r)}}_{0:1;D';\cdots;D^{(r)}} \left( \begin{matrix} [m+\alpha/k: h/k, m_1,\ldots,m_r]: \text{———}; \\ \text{———}: \; [\alpha/k: h/k]; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):1];\cdots;[(b^{(r)}):1]; \\ [(d'):1];\cdots;[(d^{(r)}):1]; \end{matrix} \; \Xi_0, \Xi_1, \ldots, \Xi_r \right), \qquad k \neq 0,$$

where $h/k > 0$, $\Delta_r$ is given by (8), and $\Xi_0, \Xi_1, \ldots, \Xi_r$ are defined by (19).

Obviously, this last formula (20) generates the product of $r$ generalized hypergeometric polynomials; it is a generalization of several known results due to Srivastava and Singhal [15].

For special values of the parameters, the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, h, p, k)$ can be reduced to the classical Hermite and Laguerre polynomials and their various generalizations studied in the literature (*cf.* [14, p. 76]). Furthermore, the generalized hypergeometric polynomials occurring in (20) can be specialized to several important classes of hypergeometric polynomials including, for example, the classical Hermite polynomials and their such generalizations as those considered by Gould and Hopper [3, p. 58]

$$\text{(21)} \qquad g_n^m(x,\lambda) = \sum_{j=0}^{[n/m]} \frac{n!}{j!(n-mj)!} \lambda^j x^{n-mj}$$

$$= x^n \, {}_mF_0 \left[ \begin{matrix} \Delta(m; -n); \\ \text{———}; \end{matrix} \; \lambda \left( -\frac{m}{x} \right)^m \right],$$

and by Brafman [1, p. 186]

$$\text{(22)} \qquad \mathscr{B}_n^m[\alpha_1,\ldots,\alpha_r; \beta_1,\ldots,\beta_s: x]$$

$$= {}_{m+r}F_s \left[ \begin{matrix} \Delta(m; -n), & \alpha_1,\ldots,\alpha_r; \\ & \beta_1,\ldots,\beta_s; \end{matrix} \; x \right],$$

where, as in (20), $\Delta(m; -n)$ abbreviates the array of $m$ parameters

$$(-n+j-1)/m, \qquad j = 1,\ldots,m,$$

$m$ being an arbitrary positive integer. The details involved in these derivations of known or new multilinear generating functions from (20) may be left as an exercise to the interested reader.

Yet another interesting application of our theorem would result when in (18) we set

$$\begin{cases} h = p = 1, \quad A = B^{(i)} = C = D^{(i)} - 1 = 0, \\ d_1^{(i)} = 1 + \beta_i, \quad \delta_1^{(i)} = s_i, \quad m_i = 1, \qquad i = 1,\ldots,r, \end{cases}$$

replace $\alpha$ by $\alpha + 1$, and $y_i$ by $y_i^{s_i}$, $i = 1,\ldots,r$, and appeal to the relationship (6) and to the explicit representation (2). We thus obtain our desired multilinear generating function for the Konhauser biorthogonal polynomials in the form:

$$(23) \qquad \sum_{n_1,\ldots,n_r=0}^{\infty} (m + n_1 + \cdots + n_r)!\, Y^{\alpha}_{m+n_1+\cdots+n_r}(x; k)$$
$$\cdot \prod_{i=1}^{r} \left\{ Z^{\beta_i}_{n_i}(y_i; s_i) \frac{u_i^{n_i}}{(1 + \beta_i)_{s_i n_i}} \right\}$$
$$= \left( \frac{\alpha + 1}{k} \right)_m \quad e^{\lambda} \Delta_r^{-m-(\alpha+1)/k}$$
$$\cdot F^{1:0;\cdots;0}_{0:1;\cdots;1}\left( \begin{array}{l} [m + (\alpha + 1)/k : 1/k, 1,\ldots,1] : \\ \overline{\qquad\qquad\qquad\qquad} : \end{array} \right.$$
$$\begin{array}{ll} \overline{\qquad\qquad}; & \overline{\qquad\qquad}; \cdots; \\ [(\alpha + 1)/k : 1/k]; & [1 + \beta_1 : s_1]; \cdots; \end{array}$$
$$\left. \begin{array}{l} \overline{\qquad\qquad}; \\ [1 + \beta_r : s_r]; \end{array} -\frac{x}{\Delta_r^{1/k}}, -\frac{u_1 y_1^{s_1}}{\Delta_r}, \ldots, -\frac{u_r y_r^{s_r}}{\Delta_r} \right),$$

where, by definition,

$$(24) \qquad \alpha > -1; \quad \beta_i > -1; \quad k, s_i = 1, 2, 3, \ldots; \qquad \forall\, i \in \{1,\ldots,r\}.$$

A seriously erroneous version of a *special* case of the multilinear generating function (23), when $s_1 = \cdots = s_r = s$, was proven earlier by Patil and Thakare [6] who incidentally used a markedly different method. In fact, (23) with $k = s_1 = \cdots = s_r = 1$ is a well-known result (involving the classical Laguerre polynomials) due to Srivastava and Singhal [15, p. 1239, Eq. (5)].

Since $s_1,\ldots,s_r$ are, by definition, positive integers, the multilinear generating function (23) would follow also as an obvious special case of (20).

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Received December 22, 1982. Supported, in part, by NSERC (Canada) Grant A-7353.

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